A point-free approach to L-Surjunctivity and Stable Finiteness

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Abstract

The category of quasi frames (or qframes) is introduced and studied. In the context of qframes we can jointly study problems related to the L-Surjunctivity and Stable Finiteness Conjectures. As a consequence of our main results, we can generalize some of the known results on these conjectures. In particular, let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$ and let N be a right R-module. It is proved that: (1) the endomorphism ring $\mathrm{End}_{R*G}(N \otimes_R R{*}G)$ is stably finite, provided N is finitely generated and has Krull dimension; (2) any linear cellular automaton $\phi : N^G \to N^G$ is surjunctive, provided N is Artinian.

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Contents

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1 Introduction

In this paper we describe a *point-free strategy* to partially solve two classical problems about the representations of a given group G. Let us introduce first the problems we are interested in, and then give an idea of our approach to their solution.

(Linear) Surjunctivity Conjecture. A map is *surjunctive* if it is non-injective or surjective. Let A be a finite set and equip $A^G = \prod_{g\in G} A$ with the product of the discrete topologies on each copy of A. There is a canonical left action of G on A^G defined by

$$gx(h) = x(g^{-1}h) \text{ for all } g, h \in G \text{ and } x \in A^G .$$

A long standing open problem by Gottschalk [19] is that of determining whether or not any continuous and G-equivariant map $\phi : A^G \to A^G$ is surjunctive, we refer to this problem as the *Surjunctivity Conjecture*. When G is amenable this problem has been known for a long time to have a positive solution but it was just in 1999 when Gromov [20] came out with a general theorem solving the problem in the positive for the large class of sofic groups (see also [30]). The general case remains open.

An analogous problem is as follows. Let $\mathbb{K}$ be a field, let V be a finite dimensional $\mathbb{K}$-vector space, endow V^G with the product of the discrete topologies and consider the canonical left G-action on V^G. The *L-Surjunctivity Conjecture* states that any G-equivariant continuous and $\mathbb{K}$-*linear* map $V^G \to V^G$ is surjunctive. This conjecture is known for the case of sofic groups, that follows again by Gromov's general surjunctivity theorem in [20] (see also [10]). Again, the general case is unknown.

Stable Finiteness Conjecture. A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Furthermore, R is *stably finite* if the ring of square $k \times k$ matrices $\mathrm{Mat}_k(R)$ is directly finite for all $k \in \mathbb{N}_+$. A long-standing open problem due to Kaplansky [21] is to determine whether the group ring $\mathbb{K}[G]$ is stably finite for any field $\mathbb{K}$, we refer to this problem as the *Stable Finiteness Conjecture*.
Notice that $\mathrm{Mat}_k(\mathbb{K}[G]) \cong \mathrm{End}_{\mathbb{K}[G]}(\mathbb{K}[G]^k)$, so an equivalent way to state the Stable Finiteness Conjecture is to say that any surjective endomorphism of a free right (or left) $\mathbb{K}[G]$-module of finite rank is injective. In case the field $\mathbb{K}$ is commutative and has characteristic 0, then the problem was solved in the positive by Kaplansky. There was no progress in the positive characteristic case until 2002, when Ara, O'Meara and Perera [5] proved that a group algebra $D[G]$ is stably finite whenever G is residually amenable and D is any division ring. This last result was generalized by Elek and Szabó [17], that proved the same result for G a sofic group (see also [10] and [6] for alternative proofs). We remark that in [5] one can also find a proof of the fact that any crossed product $D{*}G$ (see Section 5.1) of a division ring D and an amenable group G is stably finite.

Relations between the conjectures. In the introduction of [17], it is observed that the Surjunctivity Conjecture implies the Stable Finiteness Conjecture, in case $\mathbb{K}$ is a finite field. Roughly speaking, the idea is to consider $\mathbb{K}$ as a discrete compact Abelian group, view $(\mathbb{K}[G])^k$ as a dense subgroup of the compact group $(\mathbb{K}^k)^G$ and extend maps by continuity. Let us give a different

argument. In brief, consider the finite field $\mathbb{K}$ as a finite discrete Abelian group; then, applying Pontryagin-Van Kampen's duality to a G-equivariant endomorphism ϕ of the discrete group $(\mathbb{K}^k)^{(G)}$ (with the right G-action) we get a continuous G-equivariant endomorphism $\widehat{\phi}$ of the compact group $(\mathbb{K}^k)^G$ (with the left G-action) endowed with the product of the discrete topologies, and viceversa. In fact, Pontryagin-Van Kampen's duality induces an anti-isomorphism between the ring of G-equivariant $\mathbb{K}$-endomorphisms of $(\mathbb{K}^k)^{(G)}$ and the ring of G-equivariant continuous $\mathbb{K}$-endomorphisms of $(\mathbb{K}^k)^G$. Ceccherini-Silberstein and Coornaert [10] give a different argument that shows that the same ring anti-isomorphism holds for arbitrary fields (they compose their map with the usual anti-involution on $\mathrm{Mat}_k(\mathbb{K}[G])$ to make it an actual ring isomorphism). This proves that the L-Surjunctivity Conjecture is equivalent to the Stable Finiteness Conjecture. The Appendix of this paper is devoted to recall the classical theory of duality between categories of discrete and linearly compact modules. As a consequence we can generalize the ring anti-isomorphism described above (see Corollary A.15), clarifying the relation between Stable Finiteness and L-Surjunctivity Conjectures.

Our point-free approach. Let G be a group, let R be a ring and fix a crossed product $R{*}G$. Let N be a right R-module, let $M_{R*G} = N \otimes_R R{*}G$ and consider an endomorphism $\phi : M \to M$. It is well known that the poset $\mathcal{L}(M)$ of R-submodules of M (ordered by inclusion) is a lattice with very good properties, furthermore, ϕ induces a semi-lattice homomorphism $\Phi : \mathcal{L}(M) \to \mathcal{L}(M)$, that associates to a submodule $K \leqslant M$ the submodule $\phi(K)$. There is a natural right action of G on $\mathcal{L}(M)$, induced by the $R{*}G$-module structure of M (even if there is no natural G-action on M in general), and Φ is G-equivariant with respect to this action. It turns out that ϕ is injective (resp., surjective) if and only if Φ has the same property. Using this construction we can translate (a general form of) the Stable Finiteness Conjecture in terms of some "well-behaved" lattices with a G-action and semi-lattice G-equivariant endomorphisms on them.

Similarly, consider a left R-module N, take the product N^G endowed with the product of the discrete topologies and the usual left G-action, and consider a G-equivariant continuous endomorphism $\phi : N^G \to N^G$. If N is linearly compact in the discrete topology (e.g., it is Artinian), one can show that the poset $\mathcal{N}(N^G)$ of closed submodules of N^G, ordered by *reverse* inclusion, is a lattice with many common features with a lattice of submodules of a discrete module. Furthermore, ϕ induces a semi-lattice homomorphism $\Phi : \mathcal{N}(N^G) \to \mathcal{N}(N^G)$, that associates to a closed submodule $K \leqslant N^G$ its preimage $\phi^{-1}(K)$. There is a natural right action of G on $\mathcal{N}(N^G)$, induced by the left G-action on N^G, and Φ is G-equivariant with respect to this action. It turns out that ϕ is injective (resp., surjective) if and only if Φ is surjective (resp., injective). Thus, with this construction we can translate (a general form of) the L-Surjectivity Conjecture in terms of lattices with a G-action and G-equivariant semi-lattice endomorphisms on them, exactly as we did for the Stable Finiteness Conjecture.

In Section 2, we introduce the category QFrame of *quasi frames* (or *qframes*). Roughly speaking, a qframe is a lattice with properties analogous to a lattice of sub-modules. We study many constructions in that category, mimicking similar constructions on modules.

In Section 3 we introduce and study two cardinal invariants attached to a qframe: its Krull and its Gabriel dimension. In the final part of the section we introduce the concepts of torsion and localization of a qframe.

In Section 4, we prove a general theorem for a G-equivariant endormorphism of qframes, where G is a sofic group. We work first in semi-Artinian qframes (that are exactly the qframes with Gabriel dimension 1), see Theroem 4.5, and then, using torsion and localization, we lift our result to higher Gabriel dimension, see Theorem 4.7.

In Section 5, we apply the general theorem to the above conjectures. In particular, we can

prove a general version of the L-Surjunctivity Conjecture for sofic groups:

Theorem 5.12 *Let R be a ring, let G be a sofic group and let ${}_RN$ be a left R-module. If ${}_RN$ is Artinian, then any G-equivariant continuous morphism $\phi : N^G \to N^G$ is surjunctive.*

Notice that the above theorem generalizes in different directions the main result of [12] and [11]. Furthermore, we prove a general version of the Stable Finiteness Conjecture in the sofic case, generalizing results of [17] and [5]:

Theorem 5.3 *Let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$, let N_R be a finitely generated right R-module and let $M_{R*G} = N \otimes_R R{*}G$.*

(1) *If N_R is Noetherian, then any surjective homomorphism $\phi : M \to M$ is injective;*

(2) *If N_R is finitely generated and has Krull dimension, then* $\mathrm{End}_{R*G}(N \otimes_R R{*}G)$ *is stably finite.*

As a consequence of the above results we obtain that both the L-Srjunctivity and the Stable Finiteness Conjectures hold for free-by-sofic and for (finite-by-polycyclic)-by-sofic groups.

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2 Quasi-frames

In this section we introduce the category of quasi-frames and we introduce the technical machinery that will be used in the proof of our main results.

2.1 The category of quasi-frames

Recall that a poset $(L, \leqslant)$ is a *lattice* if any finite subset $F = \{f_1, \dots, f_k\} \subseteq L$ has a *least upper bound* (also called the *join of* F), denoted by $\bigvee F$ or $f_1 \vee \cdots \vee f_k$, and a *greatest lower bound* (also called the *meet* of F), denoted by $\bigwedge F$ or $f_1 \wedge \cdots \wedge f_k$.

Given a lattice $(L, \leqslant)$ and two elements x, $y \in L$, the *segment* between x and y is

$$[x, y] = \{s \in L : x \leqslant s \leqslant y\} .$$

We also let $(x, y] = [x, y]\backslash\{x\}$, $[x, y) = [x, y]\backslash\{y\}$ and $(x, y) = [x, y]\backslash\{x, y\}$. Notice that $[x, y]$ is itself a lattice with the partial order induced by L.

Let us recall the following properties that a lattice $(L, \leqslant)$ may (or may not) have:

(1) $(L, \leqslant)$ is *bounded* if it has a maximum (denoted by 1) and a minimum (denoted by 0);

(2) $(L, \leqslant)$ is *modular* if, for all a, b and $c \in L$ with $a \leqslant c$,

$$a \vee (b \wedge c) = (a \vee b) \wedge c ;$$

(3) $(L, \leqslant)$ is *distributive* if, for all a, b and $c \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \;\text{ and }\; a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) ;$$

(4) $(L, \leqslant)$ is *complete* if it has joins and meets for any of its subsets (finite or infinite). By convention we put $\bigvee \varnothing = 0$;

(5) $(L, \leqslant)$ is *upper-continuous* if it is complete and, for any directed subset $\{x_i : i \in I\}$ of L (or, equivalently, for any chain in L) and any $x \in L$,

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i)\,;$$

(6) given $a \in L$, and element c is a *pseudo-complement* for a if it is maximal with respect to the property that $a \wedge c = 0$. $(L, \leqslant)$ is *pseudo-complemented* if, for any choice of $a \leqslant b \leqslant c$ in L, there is a pseudo-complement of b in the lattice $[a, c]$;

(7) $(L, \leqslant)$ is a *frame* if it is complete and, for any subset $\{x_i : i \in I\}$ of L and any $x \in L$,

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i)\,;$$

(8) $(L, \leqslant)$ is *compact* if it has a maximum $1 \in L$ and, for any subset $S \subseteq L$ such that $\bigvee S = 1$, there exists a finite subset $F \subseteq S$ such that $\bigvee F = 1$.

If a lattice is distributive, then it is also modular, an upper-continuous modular lattice is always pseudo-complemented, furthermore, any complete lattice is bounded. Notice also that a frame has all the properties listed in (1)–(6).

Example 2.1. (1) Let X be a set. The power set $\mathcal{P}(X)$, ordered by inclusion, is a frame;

(2) the family of open sets $\mathrm{Open}(X)$ of a topological spaces (X, τ), ordered by inclusion, is a frame;

(3) given a ring R and a right R-module M, the family $\mathcal{L}(M)$ of submodules of M, ordered by inclusion, is an upper-continuous modular lattice. Furthermore, $\mathcal{L}(M)$ is compact if and only if M is finitely generated.

Let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be two lattices and consider a map $\phi : L_1 \to L_2$ between them. Then,

(1) ϕ is a *semi-lattice homomorphism* provided $\phi(x \vee y) = \phi(x) \vee \phi(y)$, for all x and $y \in L_1$;

(2) ϕ is a *lattice homomorphism* if it is a semi-lattice homomorphism and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, for all x and $y \in L_1$;

(3) ϕ *commutes with arbitrary joins* if, for any subset S of L_1, $\phi(\bigvee S) = \bigvee_{s \in S} \phi(s)$, meaning that, if $\bigvee S$ exists, then also $\bigvee_{s \in S} \phi(s)$ exists and it coincides with $\phi(\bigvee S)$;

(4) ϕ *preserves segments* if $\phi([a, b]) = [\phi(a), \phi(b)]$ for all $x \leqslant y \in L_1$.

Notice that, if L_1 has a minimum element $0 = \bigvee \varnothing$ and ϕ commutes with arbitrary joins, then $\phi(0) = \bigvee \varnothing = 0 \in L_2$ is a minimum element in L_2. Furthermore, a map which preserves segments is surjective if and only if its image contains 0 and 1.

Example 2.2. Let R be a ring and consider a homomorphism of right R modules $\phi : M \to N$. Then, ϕ induces a map

$$\Phi : \mathcal{L}(M) \to \mathcal{L}(N)\,,$$

such that $\Phi(K) = \phi(K) \leqslant N$, for all $K \in \mathcal{L}(M)$. One can show that Φ is a semi-lattice homomorphism that commutes with arbitrary joins and that preserves segments, while in general Φ is not a lattice homomorphism.

Definition 2.3. *A* quasi-frame *(or* qframe*) is an upper-continuous modular lattice. A map between two quasi-frames is a* homomorphism of quasi-frames *if it is a homomorphism of semi-lattices that preserves segments and commutes with arbitrary joins. We denote by* QFrame *the category of quasi-frames and homomorphisms of quasi-frames with the obvious composition.*

Given a qframe $(L, \leqslant)$ and $x \in L$, a segment of the form $[0, x]$ is said to be a *sub-qframe* of L.

2.2 Constructions in QFrame

Definition 2.4. *Let $\phi : L_1 \to L_2$ be homomorphism of qframes. The element*

$$\mathrm{Ker}(\phi) = \bigvee_{\phi(x)=0} x \in L_1$$

is said to be the kernel of ϕ. *We say that ϕ is* algebraic *provided the restriction $\phi : [\mathrm{Ker}(\phi), 1] \to L_2$ of ϕ to $[\mathrm{Ker}(\phi), 1]$ is injective.*

Notice that an algebraic homomorphism of qframes is injective if and only if its kernel is 0. It is a useful exercise to prove that the morphism Φ in Example 2.2 is algebraic. Let us remark that, as noticed in [2, Proposition 0.8], there is quite a strong relation between algebraic homomorphisms of qframes and linear morphisms as introduced in [1]. In [2, Example 0.9] one can also find an easy example of a non-algebraic homomorphism of qframes.

Definition 2.5. *Let $(L, \leqslant)$ be a qframe, let I be a set and consider a subset $\mathcal{F} = \{x_i : i \in I\} \subseteq L$ such that $x_i \neq 0$ for all $i \in I$. We say that $\mathcal{F}$ is a* join-independent family *if, for any $i \in I$,*

$$\left(\bigvee_{j \in I \setminus \{i\}} x_j \right) \wedge x_i = 0\,.$$

Furthermore $\mathcal{F}$ is a basis *for L if it is join-independent and $\bigvee_{i \in I} x_i = 1$.*

As an example one can consider a family $\{M_i : i \in I\}$ of right R-modules and the direct sum $M = \bigoplus_I M_i$. Then, identifying each M_i with a submodule of M in the obvious way, the family $\{M_i : i \in I\}$ is a basis of the qframe $\mathcal{L}(M)$.

The following lemma will be useful later on.

Lemma 2.6. *Let $(L, \leqslant)$ be a qframe, let $x \in L$ and let $\{y_i : i \in I\}$ be a basis of L. Then,*

(1) *if $x \neq 0$, there exists a finite subset of I such that $x \wedge \bigvee_{i \in F} y_i \neq 0$;*

(2) *if $[0, x]$ is compact, there exists a finite subset F of I such that $x \leqslant \bigvee_{i \in F} y_i$.*

Proof. (1) Notice that $0 \neq x = x \wedge \bigvee\{\bigvee_{i \in F} y_i : F \subseteq I \text{ finite}\} = \bigvee\{x \wedge \bigvee_{i \in F} y_i : F \subseteq I \text{ finite}\}$, so for at least one finite subset F of I, $x \wedge \bigvee_{i \in F} y_i \neq 0$.
(2) Notice that $x = x \wedge \bigvee_{i \in I}\{\bigvee_{i \in G} y_i : G \subseteq I \text{ finite}\} = \bigvee\{x \wedge \bigvee_{i \in G} y_i : G \subseteq I \text{ finite}\}$. By the definition of compact lattice, there exists a finite subset K of the set of finite subsets of I such that $x = \bigvee\{x \wedge \bigvee_{i \in G} y_i : G \in K\}$. Taking $F = \bigcup_{G \in K} G$ we get

$$x = \bigvee \left\{ x \wedge \bigvee_{i \in G} y_i : G \in K \right\} \leqslant x \wedge \bigvee_{i \in F} y_i \leqslant x\,.$$

Thus, $x = x \wedge \bigvee_{i \in F} y_i$, which means exactly that $x \leqslant \bigvee_{i \in F} y_i$. ☐

Definition 2.7. *Let I be a set and, for all $i \in I$, let $(L_i, \leqslant)$ be a qframe. The* product *of this family is*

$$\prod_{i \in I} L_i = \{\underline{x} = (x_i)_I : x_i \in L_i\,,\ \textit{for all } i \in I\}$$

with the partial order relation defined by

$$(\ \underline{x} \leqslant \underline{y}\ ) \iff (\ x_i \leqslant y_i\,,\ \textit{for all } i \in I\ )\,.$$

One can verify that $(\prod_{i \in I} L_i, \leqslant)$ is a qframe. Furthermore, for any subset $J \subseteq I$ the usual projection $\pi_J : \prod_{i \in I} L_i \to \prod_{j \in J} L_j$ is a surjective homomorphism of qframes, and the usual inclusion $\epsilon_J : \prod_{j \in J} L_j \to \prod_{i \in I} L_i$ is an injective homomorphism of qframes.

We conclude this subsection describing the quotient objects in the category of qframes.

Definition 2.8. *A* congruence *on a qframe $(L, \leqslant)$ is a subset $\mathcal{R} \subseteq L \times L$ that satisfies the following properties:*

(Cong.1) *$\mathcal{R}$ is an equivalence relation;*

(Cong.2) *for all a, b and $c \in L$, $(a, b) \in \mathcal{R}$ implies $(a \vee c, b \vee c) \in \mathcal{R}$;*

(Cong.3) *for all a, b and $c \in L$, $(a, b) \in \mathcal{R}$ implies $(a \wedge c, b \wedge c) \in \mathcal{R}$.*

Furthermore, if $\mathcal{R}$ satisfies the following condition (Cong.4), then $\mathcal{R}$ is a strong congruence*:*

(Cong.4) *for all $a \in L$, the equivalence class $[a] = \{b \in L : (a, b) \in \mathcal{R}\}$ has a maximum.*

Usually, given a congruence $\mathcal{R}$ on a qframe $(L, \leqslant)$, we write $a \sim b$ to denote that $(a, b) \in \mathcal{R}$.

The following lemma is analogous to [4, Proposition 2.2], anyway we prefer to give a complete proof to make the paper reasonably self-contained.

Lemma 2.9. *Let $(L, \leqslant)$ be a qframe and let $\mathcal{R}$ be a strong congruence on L. Let $L/\mathcal{R}$ be the set of equivalence classes in L and endow it with the following binary relation:*

$$(\ [a] \preceq [b]\ ) \iff (\ \exists a' \in [a] \textit{ and } b' \in [b] \textit{ such that } a' \leqslant b'\ )\,.$$

Then, $\preceq$ is a partial order and $(L/\mathcal{R}, \preceq)$ is a qframe. Furthermore, the canonical map $\pi : L \to L/\mathcal{R}$ such that $x \mapsto [x]$ is a surjective homomorphism of qframes.

Proof. *$\preceq$ is a partial order.* The unique non-obvious think is to verify that $[a] \preceq [b] \preceq [c] \in L/\mathcal{R}$ implies that $[a] \preceq [c]$. To do so, take $a' \in [a]$, b' and $b'' \in [b]$, and $c' \in [c]$ such that $a' \leqslant b'$ and $b'' \leqslant c'$. It is then clear that $a' \leqslant b' \vee b'' \leqslant b' \vee c'$, and also that $c' = c' \vee b'' \sim c' \vee b'$, thus $[a] \preceq [c]$.
Lattice structure. Let a and $b \in L$ and let us show that $[a \wedge b]$ is a greatest lower bound for $[a]$ and $[b]$ in $L/\mathcal{R}$. Indeed, it is clear that $[a \wedge b]$ is $\preceq$ of both $[a]$ and $[b]$. Furthermore, given $c \in L$ such that $[c] \preceq [a]$ and $[c] \preceq [b]$, there exit $a' \in [a]$, $b' \in [b]$ and c', $c'' \in [c]$, such that $c' \leqslant a'$ and $c'' \leqslant b'$. Thus, $[c] = [c' \wedge c''] \preceq [a' \wedge b'] = [a \wedge b]$. One can show analogously that $[a \vee b]$ is a least upper bound for $[a]$ and $[b]$.
Modularity. Let a, b and $c \in L$ be elements and suppose $[a] \preceq [c]$. Choose $a' \in [a]$ and $c' \in [c]$ such that $a' \leqslant c'$, then, by the modularity of L, we have that $a' \vee (b \wedge c') = (a' \vee b) \wedge c'$ which implies

$$[a] \vee ([b] \wedge [c]) = [a'] \vee ([b] \wedge [c']) = ([a'] \vee [b]) \wedge [c'] = ([a] \vee [b]) \wedge [c]\,.$$

Completeness. Consider a family $\mathcal{F} = \{[x_i] : i \in I\}$ in $L/\mathcal{R}$, we claim that $[\bigvee_{i \in I} x_i]$ is a least upper bound for $\mathcal{F}$. In fact, it is clear that $[\bigvee_{i \in I} x_i] \succeq [x_j]$ for all $j \in I$. Furthermore, given $c \in L$ such

that $[c] \succeq [x_i]$ for all $i \in I$, we can choose $x_i' \in [x_i]$ such that $x_i' \leqslant \bar{c}$ for all $i \in I$, where $\bar{c}$ is the maximum of $[c]$, for all $i \in I$. Letting $\bar{x}_i$ be the maximum of $[x_i]$, for all $i \in I$, $\bar{c} = \bar{c} \vee x_i' \sim \bar{c} \vee \bar{x}_i$ and so $\bar{x}_i \leqslant \bar{c}$, for all $i \in I$. Thus, $[c] = [\bar{c}] \succeq [\bigvee_{i\in I} \bar{x}_i] \succeq [\bigvee_{i\in I} x_i]$.
$(L/\mathcal{R}, \preceq)$ is a qframe. We have just to verify upper continuity. Let $\{[x_i] : i \in I\}$ be a directed family in $L/\mathcal{R}$ and let $\bar{x}_i = \bigvee [x_i]$, for all $i \in I$. The set $\{\bar{x}_i : i \in I\}$ is directed and so, for all $x \in L$, $x \wedge \bigvee_{i\in I} \bar{x}_i = \bigvee_{i\in I}(x \wedge \bar{x}_i)$. Thus, by our description of lattice operations,

$$[x] \wedge \bigvee_{i\in I}[x_i] = [x] \wedge \bigvee_{i\in I}[\bar{x}_i] = \bigvee_{i\in I}([x] \wedge [\bar{x}_i]) = \bigvee_{i\in I}([x] \wedge [x_i])\,.$$

π is a surjective homomorphism of qframes. It is all clear from the description of the lattice operation in $L/\mathcal{R}$ a part the fact that π preserves segments. So take $x \leqslant y \in L$ and consider $[z] \in [[x], [y]]$. Let $x' \in [x]$ and $z' \in [z]$ be such that $x' \leqslant z'$. Clearly, $x \leqslant z' \vee x \in [z]$, in fact, $x \sim x'$ implies $z' \vee x \sim z' \vee x' = z'$. Furthermore, $y \geqslant (z' \vee x) \wedge y \in [z]$, in fact, given $z'' \in [z]$ and $y' \in [y]$ such that $z'' \leqslant y'$, we obtain $(z' \vee x) \wedge y \sim z'' \wedge y \sim z'' \wedge y' = z''$. Thus, $(z' \vee x) \wedge y \in [x, y]$ and $\pi((z' \vee x) \wedge y) = [z]$. ☐

2.3 Composition length

Let $(L, \leqslant)$ be a qframe. Given a finite chain

$$\sigma : x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n$$

of elements of L, we say that the length $\ell(\sigma)$ of σ is the number of strict inequalities in the chain.

Definition 2.10. *Let $(L, \leqslant)$ be a qframe. The* length *of L is*

$$\ell(L) = \sup\{\ell(\sigma) : \ \sigma \text{ a finite chain of elements of } L\} \in \mathbb{N} \cup \{\infty\}\,.$$

For any element $x \in L$ we use the notation $\ell(x)$ to denote the length of the segment $[0, x]$.

A qframe $(L, \leqslant)$ is said to be *trivial* if it has just one element. In what follows, by *non-trivial* qframe we mean a qframe which contains at least two elements. Furthermore, $(L, \leqslant)$ is said to be an *atom* (or to be *simple*) if it has two elements.

Remark 2.11. A qframe $(L, \leqslant)$ is trivial if and only if $\ell(L) = 0$, while it is an atom if and only if $\ell(L) = 1$.

Definition 2.12. *Let $(L, \leqslant)$ be a qframe and consider a finite chain*

$$\sigma : 0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = 1\,.$$

If all the segments $[x_i, x_{i+1}]$, with $i = 0, \ldots, n-1$, are simple, then we say that σ is a composition series.

The following lemma is known as Artin-Schreier's Refinement Theorem.

Lemma 2.13. [28, Proposition 3.1, Ch. III] *Let $(L, \leqslant)$ be a qframe, let $a \leqslant b \in L$ and let*

$$\sigma_1 : a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = b \quad \text{and} \quad \sigma_2 : a = y_0 \leqslant y_1 \leqslant \cdots \leqslant y_m = b\,.$$

Then, there exists a series $\sigma : a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_t = b$ which is equivalent to a refinement of both σ_1 and σ_2.

Using Lemma 2.13, one can deduce the following lemma, which is usually known as Jordan-Hölder Theorem.

Lemma 2.14. *Let $(L, \leqslant)$ be a qframe of finite length. Then,*

(1) *any finite chain in L can be refined to a composition series;*

(2) *any two composition series in L have the same length;*

(3) *$\ell(L) = n$ if and only if there exists a composition series of length n in L.*

Definition 2.15. *A qframe $(L, \leqslant)$ is*

– Noetherian *if any ascending chain $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \ldots$ stabilizes at some point;*

– Artinian *if any descending chain $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant \ldots$ stabilizes at some point.*

Using Lemma 2.14, one can prove that $\ell(L) < \infty$ if and only if L is both Noetherian and Artinian (see also Lemma 2.25). The following lemma is well-known.

Lemma 2.16. *Let $(L, \leqslant)$ be a qframe. Then, L is Noetherian if and only if $[0, x]$ is compact for all $x \in L$.*

Lemma 2.17. [15, Lemma 3.2] *Let $(L, \leqslant)$ be a qframe of finite length and let x, $y \in L$. Then,*

$$\ell(x \vee y) + \ell(x \wedge y) = \ell(x) + \ell(y) .$$

Lemma 2.18. *Let $\phi : L_1 \to L_2$ be a homomorphism of qframes:*

(1) *if ϕ is injective, then $\ell(L_1) \leqslant \ell(L_2)$;*

(2) *if ϕ is surjective, then $\ell(L_2) \leqslant \ell(L_1)$.*

Proof. (1) Let $\sigma :\ x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be a chain in L_1, then $\phi(\sigma) :\ \phi(x_1) \leqslant \phi(x_2) \leqslant \cdots \leqslant \phi(x_n)$ is a chain in L_2. Furthermore, if $x_i \neq x_j$, then $\phi(x_i) \neq \phi(x_j)$ by injectivity. Thus, $\ell(\phi(\sigma)) = \ell(\sigma)$ and so $\ell(L_1) \leqslant \ell(L_2)$.
(2) Let $\sigma :\ x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be a chain in L_2. Since ϕ is surjective, there exist $y_1, \ldots, y_n \in L_1$ such that $\phi(y_i) = x_i$ for all $i = 1, \ldots, n$. Clearly, $\sigma' :\ y_1 \leqslant (y_1 \vee y_2) \leqslant \cdots \leqslant (y_1 \vee y_2 \vee \ldots \vee y_n)$ and, for all $i = 1, \ldots, n$, $\phi(y_1 \vee \ldots \vee y_i) = \phi(y_1) \vee \ldots \vee \phi(y_i) = x_1 \vee \ldots \vee x_i = x_i$. If $x_i \neq x_{i+1}$, then $y_1 \vee \ldots \vee y_i \neq y_1 \vee \ldots \vee y_i \vee y_{i+1}$ and so $\ell(\sigma) \leqslant \ell(\sigma')$. Thus, $\ell(L_2) \leqslant \ell(L_1)$. ☐

Corollary 2.19. *Let I be a set. For all $i \in I$, let $(L_i, \leqslant)$ be a non-trivial qframe and let $L = \prod_I L_i$. Then,*

$$\ell(L) = \begin{cases} \sum_{i \in I} \ell(L_i) & \textit{if } I \textit{ is finite;} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. If $\ell(L_i) = \infty$ for some $i \in I$ there is nothing to prove, so we suppose that $\ell(L_i)$ is finite for all $i \in I$. Let $\epsilon_i : L_i \to L$ be the canonical inclusion and let $1_i = \bigvee \epsilon(L_i)$, for all $i \in I$. Notice that $\epsilon_i(L_i) = [0, 1_i]$, so $\ell(L_i) = \ell(1_i)$, and $L = [0, \bigvee_{i \in I} 1_i]$, so $\ell(L) = \ell(\bigvee_{i \in I} 1_i)$.
When I is finite, the proof follows by Lemma 2.17 and the fact that, $1_i \wedge \bigvee_{j \neq i} 1_j = 0$.
If I is not finite, then for any finite subset $J \subseteq I$, we have $\ell(\prod_J L_j) = \sum_J \ell(L_j) \geqslant |J|$ by the first part of the proof. Furthermore, $\ell(\prod_I L_i) \geqslant \ell(\prod_J L_j)$, by Lemma 2.18 applied to the maps $\pi_J : \prod_I L_i \to \prod_J L_j$. Thus, $\ell\left(\prod_I L_i\right) \geqslant \sup\{|J| : J \subseteq I \text{ finite}\} = \infty$. ☐

In the following lemma we verify that the qframes with finite length are Hopfian and coHopfian objects in QFrame.

Lemma 2.20. *Let $(L, \leqslant)$ be a qframe of finite length, let $(L', \leqslant)$ be a qframe, and let $\phi : L \to L'$ be a homomorphism of qframes. Then,*

(1) *ϕ is injective if and only if $\ell(L) = \ell(\phi(L))$;*

(2) *ϕ is surjective if and only if $\ell(\phi(L)) = \ell(L')$.*

In particular, if $\ell(L) = \ell(L')$, then ϕ is injective if and only if it is surjective.

Proof. (1) Suppose that $\ell(L) = \ell(\phi(L))$ and let $x, y \in L$ be such that $\phi(x) = \phi(y)$. If, looking for a contradiction $x \neq y$, then either $x < x \vee y$ or $y < x \vee y$. Without loss of generality, we suppose that $x < x \vee y$. Take the chain $0 \leqslant x < x \vee y \leqslant 1$ between 0 and 1 and refine it to a composition chain

$$\sigma : \ 0 \leqslant \cdots \leqslant x < \cdots < x \vee y < \cdots \leqslant 1\,,$$

thus $\ell(\sigma) = \ell(L)$ (see Lemma 2.14). The image via a homomorphism of qframes of a composition chain is a (eventually shorter) composition chain in the image. Thus, $\ell(\phi(\sigma)) = \ell(\phi(L)) = \ell(L) = \ell(\sigma)$, in particular, $\phi(x) \neq \phi(x \vee y) = \phi(x) \vee \phi(y)$, which contradicts the fact that $\phi(x) = \phi(y)$. The converse is trivial since, if ϕ is injective, then $L \cong \phi(L)$ and then clearly $\ell(L) = \ell(\phi(L))$ (use, for example, Lemma 2.18).
(2) Suppose that ϕ is not surjective and consider a composition chain $\sigma : 0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = \phi(1)$ in $\phi(L)$. We can define a longer chain $\sigma' : 0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n < 1$ in L'. Hence, $\ell(\phi(L)) = \ell(\sigma) < \ell(\sigma') \leqslant \ell(L')$. The converse is trivial since $\phi(L) = L'$ clearly implies that $\ell(\phi(L)) = \ell(L')$. □

2.4 Socle series

Definition 2.21. *Let $(L, \leqslant)$ be a qframe. The* socle *$s(L)$ of L is the join of all the atoms in L. For all $x \in L$, we let $s(x) = s([0, x])$.*

Lemma 2.22. *Let $(L, \leqslant)$ be a qframe and let I be a set. Then,*

(1) *$s(x) \leqslant x$ and $s(x_1) \leqslant s(x_2)$, for all $x \in L$ and $x_1 \leqslant x_2 \in L$;*

(2) *$s(\bigvee_{i\in I} x_i) \geqslant \bigvee_{i\in I} s(x_i)$, where $x_i \in L$ for all $i \in I$. Furthermore equality holds if $\{x_i : i \in I\}$ is join-independent;*

(3) *$s(\bigwedge_{i\in I} x_i) \leqslant \bigwedge_{i\in I} s(x_i)$, where $x_i \in L$ for all $i \in I$;*

(4) *if $\phi : L \to L'$ is a homomorphism of qframes, then $\phi(s(L)) \leqslant s(L')$.*

Proof. Parts (1) and (3) follow by the properties described in [15, page 47]. Part (2) follows by [2, Proposition 1.4]. For part (4), notice that $\phi(s(L)) = \phi(\bigvee\{x \in L : [0, x]$ is an atom$\} = \bigvee\{\phi(x) : [0, x]$ is an atom$\} \leqslant \bigvee\{y \in L' : [0, y]$ is an atom$\}$ (use the fact that ϕ takes intervals to intervals). □

Thanks to part (4) of Lemma 2.22, we can give the following

Definition 2.23. *Let $(L, \leqslant)$ be a qframe and let* $\mathrm{Soc}(L) = [0, s(L)]$. *Furthermore, given a homomorphism $\phi : L \to L'$ of qframes, we denote by* $\mathrm{Soc}(\phi) : \mathrm{Soc}(L) \to \mathrm{Soc}(L')$ *the restriction of ϕ. This defines a covariant functor* $\mathrm{Soc} : \mathrm{QFrame} \to \mathrm{QFrame}$.

It is not difficult to show that Soc is compatible with the composition of morphisms, so that the above definition is correct.

We can iterate the procedure that defines the socle as follows:

Definition 2.24. *Let $(L, \leqslant)$ be qframe. Then,*

- $s_0(L) = s(L)$;
- *for any ordinal α, $s_{\alpha+1}(L) = s([s_\alpha(L), 1])$;*
- *for any limit ordinal α, $s_\lambda(L) = \bigvee_{\alpha<\lambda} s_\alpha(L)$.*

L is semi-Artinian *if $s_\tau(L) = 1$ for some ordinal τ.*

One can show that the uniform dimension (see [15, Chapter 8]) of a semi-Artinian qframe is the length of its socle.

Lemma 2.25. [15, Theorem 5.2 and Proposition 5.3] *Let $(L, \leqslant)$ be a qframe. Then,*

(1) *L is semi-Artinian if and only if $[0, x]$ and $[x, 1]$ are semi-Artinian for all $x \in L$;*

(2) *L is semi-Artinian and Noetherian if and only if $\ell(L) < \infty$.*

3 Krull and Gabriel dimension

In this section we introduce the notions of Krull and Gabriel dimension of a qframe and we compare these two concepts. The notion of Gabriel dimension is then used to define torsion and localization endofunctors of the category of qframes.

3.1 Krull and Gabriel dimension

Definition 3.1. *Let $(L, \leqslant)$ be a qframe. The* Krull dimension $\mathrm{K.dim}(L)$ *of L is defined as follows:*

- $\mathrm{K.dim}(L) = -1$ *if and only if L is trivial;*
- *if α is an ordinal and we already defined what it means to have Krull dimension β for any ordinal $\beta < \alpha$, $\mathrm{K.dim}(L) = \alpha$ if and only if $\mathrm{K.dim}(L) \neq \beta$ for all $\beta < \alpha$ and, for any descending chain*
$$x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \geqslant x_n \geqslant \ldots$$
in L, there exists $\bar{n} \in \mathbb{N}_+$ such that $\mathrm{K.dim}([x_n, x_{n+1}]) = \beta_n$ for all $n \geqslant \bar{n}$ and $\beta_n < \alpha$.

If $\mathrm{K.dim}(L) \neq \alpha$ for any ordinal α we set $\mathrm{K.dim}(L) = \infty$.

Notice that the qframes with 0 Krull dimension are precisely the Artinian qframes.

Definition 3.2. *A subclass $\mathcal{X} \subseteq \mathrm{Ob(QFrame)}$ is a* Serre class *if it is closed under isomorphisms and, given $L \in \mathrm{Ob(QFrame)}$ and $x \leqslant y \leqslant z \in L$, $[x, y]$, $[y, z] \in \mathcal{X}$ if and only if $[x, z] \in \mathcal{X}$.*

The class of all qframes with Krull dimension $\leqslant \alpha$ for some ordinal α is a Serre class (see [15, Proposition 13.5]).

Lemma 3.3. *Let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be qframes. If $\mathrm{K.dim}(L_1)$ exists and if there exists a surjective homomorphism of qframes $\phi : L_1 \to L_2$, then $\mathrm{K.dim}(L_1) \geqslant \mathrm{K.dim}(L_2)$.*

Proof. Let us proceed by induction on $\mathrm{K.dim}(L_1) = \alpha$. If $\alpha = -1$, then clearly also $\mathrm{K.dim}(L_2) = -1$. Suppose now that $\alpha > -1$ and that we already proved our result for all $\beta < \alpha$. If $\mathrm{K.dim}(L_2) < \mathrm{K.dim}(L_1)$ there is nothing to prove, so suppose that $\mathrm{K.dim}(L_2) \not< \mathrm{K.dim}(L_1)$ and let us show that $\mathrm{K.dim}(L_2) = \mathrm{K.dim}(L_1)$. Indeed, consider a descending chain in L_2

$$x_0 \geqslant x_1 \geqslant \cdots \geqslant x_n \geqslant \ldots$$

By the surjectivity of ϕ, we can choose $y_i \in L_1$ so that $\phi(y_i) = x_i$, for all $i \in \mathbb{N}$, let also $y'_i = \bigvee_{j \geqslant i} y_j$. It is not difficult to see that

$$y'_0 \geqslant y'_1 \geqslant \cdots \geqslant y'_n \geqslant \ldots$$

and that $\phi(y'_i) = \bigvee_{j \geqslant i} \phi(y_j) = x_i$. By definition of Krull dimension, there exists $\bar{n} \in \mathbb{N}_+$ such that $\mathrm{K.dim}([y'_n, y'_{n+1}]) = \beta_n$ for all $n \geqslant \bar{n}$ and $\beta_n < \alpha$. By inductive hypothesis, $\mathrm{K.dim}([x_n, x_{n+1}]) \leqslant \mathrm{K.dim}([y'_n, y'_{n+1}]) = \beta_n$, showing that $\mathrm{K.dim}(L_2) \leqslant \alpha$, and so, $\mathrm{K.dim}(L_2) = \alpha$. $\square$

Definition 3.4. *Let $(L, \leqslant)$ be a qframe. We define the* Gabriel dimension $\mathrm{G.dim}(L)$ *of L by transfinite induction:*

- $\mathrm{G.dim}(L) = 0$ *if and only if L is trivial. A qframe S is 0-simple (or just simple) if it is an atom;*
- *let α be an ordinal for which we already know what it means to have Gabriel dimension β, for all $\beta \leqslant \alpha$. A qframe S is α-simple if, for all $0 \neq a \in S$,* $\mathrm{G.dim}([0, a]) \not\leqslant \alpha$ *and* $\mathrm{G.dim}([a, 1]) \leqslant \alpha$*;*
- *let σ be an ordinal for which we already know what it means to have Gabriel dimension β, for all $\beta < \sigma$. Then,* $\mathrm{G.dim}(L) = \sigma$ *if* $\mathrm{G.dim}(L) \not< \sigma$ *and, for all $1 \neq a \in L$, there exists $b > a$ such that $[a, b]$ is β-simple for some ordinal $\beta < \sigma$.*

If $\mathrm{G.dim}(L) \neq \alpha$ *for any ordinal α we set* $\mathrm{G.dim}(L) = \infty$.

Notice that the qframes with Gabriel dimension 1 are precisely the semi-Artinian qframes. Also the class of all qframes with Gabriel dimension $\leqslant \alpha$ for some ordinal α is a Serre class (see part (1) of Lemma 3.7). For any ordinal α, $\mathrm{G.dim}(S) = \alpha + 1$, for any α-simple qframe S.

Lemma 3.5. *Let α be an ordinal and let $(L, \leqslant)$ be an α-simple qframe. Any non-trivial sub-qframe of L is α-simple.*

Proof. We proceed by transfinite induction on α. If $\alpha = 0$, then L is an atom and there is no non-trivial sub-qframe but L itself. Let $\alpha > 0$ and choose $0 \neq b \leqslant a \in L$. By definition, $\mathrm{G.dim}([0, b]) \not\leqslant \alpha$ so, to prove that $[0, a]$ is α-simple, it is enough to show that $\mathrm{G.dim}([b, a]) \leqslant \alpha$. If $\mathrm{G.dim}([b, a]) < \alpha$ there is nothing to prove, so let us consider the case when $\mathrm{G.dim}([b, a]) \not< \alpha$. Let $a' \in (b, a]$, choose a pseudo-complement c of a in $[a', 1]$ and let $d \in [c, 1]$ be such that $[c, d]$ is β-simple for some $\beta < \alpha$. Let $b' = d \wedge a$, then $[a', b']$ is non-trivial by the maximality included in the definition of pseudo-complement, furthermore, by modularity, $[a', b']$ is isomorphic to $[c, b' \vee c]$, which is a sub-qframe of $[c, d]$. By inductive hypothesis, $[a', b']$ is β-simple. This proves that $\mathrm{G.dim}([b, a]) = \alpha$, as desired. $\square$

Theorem 3.6. *Let $(L, \leqslant)$ be a qframe. The following statements hold true:*

(1) *L has Krull dimension if and only if any segment of L has finite uniform dimension and L has Gabriel dimension;*

(2) *if L has Krull dimension, then* $\mathrm{K.dim}(L) \leqslant \mathrm{G.dim}(L) \leqslant \mathrm{K.dim}(L) + 1$*;*

(3) *if L is Noetherian, then there exists a finite chain* $0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = 1$ *such that* $[x_{i-1}, x_i]$ *is* α_i*-simple for some ordinal* α_i*, for all* $i = 1, \ldots, n$*. Furthermore,* L *has Krull dimension and* $\mathrm{G.dim}(L) = \mathrm{K.dim}(L) + 1$*.*

Proof. For (1), see Exercise (116) in [26] (an argument to solve that exercise can be found in [18]). For parts (2) and (3) see respectively [15, Theorem 13.9] and (statement and proof of) [15, Theorem 13.10]. ☐

In the following lemmas we collect some properties of Gabriel dimension. Their proof is inspired by the treatment in [26] but we prefer to give complete proofs also here.

Lemma 3.7. *Let L be a qframe with Gabriel dimension. The following statements hold true:*

(1) *if* $a \leqslant b \in L$*, then* $\mathrm{G.dim}([a, b]) \leqslant \mathrm{G.dim}(L)$*;*

(2) *if* $a \in L$*, then* $\mathrm{G.dim}(L) = \max\{\mathrm{G.dim}([0, a]), \mathrm{G.dim}([a, 1])\}$*;*

(3) *given a subset* $\mathcal{F} \subseteq L$ *such that* $\bigvee \mathcal{F} = 1$*,* $\mathrm{G.dim}(L) = \sup\{\mathrm{G.dim}([0, x]) : x \in \mathcal{F}\}$.

(4) *if L is not trivial, then* $\mathrm{G.dim}(L) = \sup\{\mathrm{G.dim}([a, b]) : [a, b]$ β*-simple for some* $\beta\}$*;*

(5) $\mathrm{G.dim}(L) \leqslant \beta + 1$*, where* $\beta = \sup\{\mathrm{G.dim}([x, 1]) : x \neq 0\}$*.*

Proof. Let $\mathrm{G.dim}(L) = \alpha$.
(1) We proceed by transfinite induction on α. If $\alpha = 0$, there is nothing to prove, as well as when $\alpha > 0$ and $\mathrm{G.dim}([a, b]) < \alpha$. Consider the case when $\alpha > 0$ and $\mathrm{G.dim}([a, b]) \nless \alpha$. Let $a' \in [a, b)$ and let us find $b' \in (a', b]$ such that $[a', b']$ is β-simple for some $\beta < \alpha$. Indeed, we consider a pseudo-complement c of b in $[a', 1]$ and we let $d \in [c, 1]$ be such that $[c, d]$ is β-simple for some $\beta < \alpha$. Let $b' = d \wedge b$. By modularity, $[a', b'] \cong [c, (d \wedge b) \vee c]$, which is a sub-qframe of $[c, d]$. By Lemma 3.5, $[a', b']$ is β-simple.
(2) Let $\beta_1 = \mathrm{G.dim}([0, a])$ and $\beta_2 = \mathrm{G.dim}([a, 1])$. By part (1), $\alpha \geqslant \max\{\beta_1, \beta_2\}$. Let us show that $\alpha \leqslant \max\{\beta_1, \beta_2\}$, that is, given $1 \neq b \in L$ we need to find $c \in (b, 1]$ such that $[b, c]$ is γ-simple for some $\gamma < \max\{\beta_1, \beta_2\}$. Indeed, given $1 \neq b \in L$, we distinguish two cases. If $a \leqslant b$, then $b \in [a, 1]$ and so there is $c \in (b, 1]$ such that $[b, c]$ is γ-simple for some $\gamma < \beta_2$. If $a \nleqslant b$, then there is $c \in [a \wedge b, a]$ such that $[b \wedge a, c]$ is γ-simple for some $\gamma < \beta_1$ and, by modularity, $[b, b \vee c] \cong [a \wedge b, c]$.
(3) Let $\sup\{\mathrm{G.dim}([0, x]) : x \in \mathcal{F}\} = \beta$. Given $1 \neq a \in L$, we have to show that there exists $b \in [a, 1]$ such that $[a, b]$ is γ-simple for some $\gamma < \beta$. By hypothesis, there exists $x \in \mathcal{F}$ such that $x \nleqslant a$. Thus, $x \neq x \wedge a \in [0, x]$ and so there exists $b' \in [x \wedge a, x]$ such that $[x \wedge a, b']$ is γ-simple for some $\gamma < \mathrm{G.dim}([0, x]) \leqslant \beta$. Let $b = b' \vee a$; by modularity $[a, b] \cong [x \wedge a, b']$ is γ-simple as desired.
(4) Consider a continuous chain in L defined as follows:

- $x_0 = 0$;
- if $\sigma = \tau + 1$ is a successor ordinal, then $x_\sigma = 1$ if $x_\tau = 1$, while x_σ is an element $\geqslant x_\tau$ such that $[x_\sigma, x_\tau]$ is β-simple for some β;
- $x_\sigma = \bigvee_{\tau < \sigma} x_\tau$ if σ is a limit ordinal.

Since we supposed that L has Gabriel dimension, then the above definition is correct and there exists an ordinal $\bar{\sigma}$ such that $x_{\bar{\sigma}} = 1$. Let us prove our statement by induction on $\bar{\sigma}$. If $\bar{\sigma} = 1$, there is nothing to prove. Furthermore, if $\bar{\sigma} = \tau + 1$, then by part (2), $\mathrm{G.dim}(L) = \max\{\mathrm{G.dim}([0, x_\tau]), \mathrm{G.dim}([x_\tau, x_{\bar{\sigma}}])$ and we can conclude by inductive hypothesis. If $\bar{\sigma}$ is a limit ordinal, one concludes similarly using part (3).

(5) It is enough to prove the statement for γ-simple qframes for all ordinals γ and then apply part (4). So, let γ be an ordinal and let L be a γ-simple qframe. Then, $\mathrm{G.dim}(L) = \gamma + 1$ and we should prove that $\sup\{\mathrm{G.dim}([x, 1]) : x \neq 0\} \geqslant \gamma$. If, looking for a contradiction, $\sup\{\mathrm{G.dim}([x, 1]) : x \neq 0\} = \beta < \gamma$ then, just by definition, L is β-simple, that is a contradiction. ☐

Corollary 3.8. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal. Then, $\mathrm{G.dim}(L) \leqslant \alpha$ if and only if, for any element $x \neq 1$, there exists $y > x$ such that $\mathrm{G.dim}([x, y]) \leqslant \alpha$.*

Proof. Let $x_0 = 0$, for any ordinal γ let $x_{\gamma+1} = 1$ if $x_\gamma = 1$, otherwise we let $x_{\gamma+1}$ be an element $> x_\gamma$ such that $\mathrm{G.dim}([x_\gamma, x_{\gamma+1}]) \leqslant \alpha$. Furthermore, for any limit ordinal λ we let $x_\lambda = \bigvee_{\gamma<\lambda} x_\gamma$. Let us prove by transfinite induction that $\mathrm{G.dim}([0, x_\gamma]) \leqslant \alpha$ for all γ, this will conclude the proof since there exists γ such that $x_\gamma = 1$. Our claim is clear when $\gamma = 0$. Furthermore, if $\gamma = \beta + 1$ and $\mathrm{G.dim}([0, x_\beta]) \leqslant \alpha$, then by Lemma 3.7 (2), $\mathrm{G.dim}([0, x_\gamma]) \leqslant \alpha$. If γ is a limit ordinal and $\mathrm{G.dim}([0, x_\beta]) \leqslant \alpha$ for all $\beta < \gamma$, one concludes by Lemma 3.7 (3). ☐

Lemma 3.9. *Let $(L, \leqslant)$ be a qframe with Gabriel dimension, let $(L', \leqslant)$ be a qframe and let $\phi : L \to L'$ be a surjective homomorphism of qframes. Then, $\mathrm{G.dim}(L') \leqslant \mathrm{G.dim}(L)$.*

Proof. Let us proceed by transfinite induction on $\mathrm{G.dim}(L)$.
If $\mathrm{G.dim}(L) = 0$, then L is a trivial as well as L', so there is nothing to prove.
Suppose now that $\mathrm{G.dim}(L) = \alpha > 0$ and that we have already verified our claim for all $\beta < \alpha$. Let first $\alpha = \gamma + 1$ be a successor ordinal and let L be γ-simple. Then, for all $0 \neq a \in L$, $\mathrm{G.dim}([a, 1]) \leqslant \gamma$ and so, by inductive hypothesis, $\mathrm{G.dim}(\phi([a, 1])) \leqslant \gamma$. By Lemma 3.7 (5), $\mathrm{G.dim}(\phi(L)) \leqslant \gamma + 1 = \alpha$.
Let now $x' \in L'$, consider the set $\mathcal{S} = \{x \in L : \phi(x) = x'\}$ and let $\bar{x} = \bigvee \mathcal{S}$, so that $\phi(\bar{x}) = \bigvee_{x\in\mathcal{S}} \phi(x) = x'$. Let also $\bar{y} \geqslant \bar{x}$ be such that $[\bar{x}, \bar{y}]$ is β-simple for some $\beta < \alpha$ and let $y' = \phi(\bar{y}) \in L'$. Then, $y' \geqslant x'$, furthermore $y' \neq x'$ (since $y' = x'$ would imply that $\bar{y} \in \mathcal{S}$, that is, $\bar{y} = \bar{x}$, which is a contradiction). By the first part of the proof, $\mathrm{G.dim}([x', y']) \leqslant \beta + 1 \leqslant \alpha$. To conclude apply Corollary 3.8. ☐

3.2 Torsion and localization

Definition 3.10. *Let $(L, \leqslant)$ be a qframe, and let α be an ordinal. We define the* α-torsion part *of L as*

$$t_\alpha(L) = \bigvee \{x \in L : \mathrm{G.dim}([0, x]) \leqslant \alpha\}\,.$$

For any given $a \in L$, we let $t_\alpha(a) = t_\alpha([0, a])$.

Lemma 3.11. *Let $(L, \leqslant)$ be a qframe, let a, $b \in L$ and let α be an ordinal. Then,*

(1) $t_\alpha(a) = a \wedge t_\alpha(1)$*;*

(2) $t_\alpha(a \vee b) \leqslant t_\alpha(a) \vee b$*, provided $a \wedge b = 0$;*

(3) $t_\alpha(a \vee b) = t_\alpha(a) \vee t_\alpha(b)$*, provided $a \wedge b = 0$;*

In particular, $t_\alpha(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} t_\alpha(x_i)$ for any join-independent set $\{x_i : i \in I\}$ in L.

Proof. (1) By definition, $t_\alpha(a) \leqslant a \wedge t_\alpha(1)$. On the other hand, by upper continuity,

$$\begin{aligned} a \wedge \bigvee\{x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\} &= \bigvee\{a \wedge x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &= \bigvee\{x \in [0,a] : \mathrm{G.dim}([0,x]) \leqslant \alpha\} = t_\alpha(a)\,. \end{aligned}$$

This works since the family $\{x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\}$ is directed by part (2) of Lemma 3.7.
(2) Let $x \in [0, a \vee b]$ be such that $\mathrm{G.dim}([0,x]) \leqslant \alpha$, then $x \vee b \in [0, a \vee b]$ and $\mathrm{G.dim}([b, x \vee b]) = \mathrm{G.dim}([b \wedge x, x]) \leqslant \mathrm{G.dim}([0,x]) \leqslant \alpha$. This shows $(*)$ below:

$$\begin{aligned} t_\alpha(a \vee b) &= \bigvee\{x \in [0, a \vee b] : \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &\overset{(*)}{\leqslant} \bigvee\{x \vee b \in [0, a \vee b] : \mathrm{G.dim}([b, x \vee b]) \leqslant \alpha\} \\ &= \bigvee\{x \in [b, a \vee b] : \mathrm{G.dim}([b,x]) \leqslant \alpha\} \\ &\overset{(**)}{=} \bigvee\{x \vee b : x \in [0,a] \text{ and } \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &\overset{\binom{*}{**}}{=} b \vee \bigvee\{x : x \in [0,a] \text{ and } \mathrm{G.dim}([0,x]) \leqslant \alpha\} = b \vee t_\alpha(a)\,, \end{aligned}$$

where $(**)$ holds since te map $x \mapsto x \vee b$ is an isomorphism between $[0,a]$ and $[b, b \vee a]$ (use the fact that $a \wedge b = 0$), and in $\binom{*}{**}$ we used upper-continuity.
(3) It is clear that $t_\alpha(b) \vee t_\alpha(a) \leqslant t_\alpha(a \vee b)$. Using twice part (2) and the modularity of L,

$$\begin{aligned} t_\alpha(b) \vee t_\alpha(a) \leqslant t_\alpha(a \vee b) \leqslant (t_\alpha(a) \vee b) \wedge (t_\alpha(b) \vee a) &= t_\alpha(a) \vee (b \wedge (t_\alpha(b) \vee a)) \\ &= t_\alpha(a) \vee ((b \wedge a) \vee t_\alpha(b)) = t_\alpha(a) \vee t_\alpha(b)\,. \end{aligned}$$

where the last equality holds since $a \wedge b = 0$.
For the last part of the statement, notice that

$$\bigvee_{i \in I} x_i = \bigvee_{F \subseteq I \text{ finite}} \left(\bigvee_{i \in F} x_i \right) .$$

Thus, using upper-continuity and part (3) of the lemma,

$$\begin{aligned} t_\alpha\left(\bigvee_{i\in I} x_i\right) &= t_\alpha(1) \wedge \bigvee_{F \subseteq I \text{ finite}} \left(\bigvee_{i \in F} x_i\right) \\ &= \bigvee_{F \subseteq I \text{ finite}} \left(t_\alpha(1) \wedge \bigvee_{i\in F} x_i\right) = \bigvee_{F \subseteq I \text{ finite}} \left(\bigvee_{i \in F} t_\alpha(x_i)\right) = \bigvee_{i \in I} t_\alpha(x_i)\,. \end{aligned}$$

□

Lemma 3.12. *Let L be a qframe, let $x \in L$ and let $\{y_s : s \in S\} \subseteq L$. Suppose that*

(1) $[0, y_s] \cong [0, y_t]$ *for all $s, t \in S$;*

(2) $[0, y_s]$ *is Noetherian for some (hence all) $s \in S$;*

(3) $\{y_s : s \in S\}$ *is a basis for L.*

Then, $\mathrm{G.dim}([0,x])$ *is a successor ordinal.*

Proof. A consequence of Theorem 3.6 (3) is that, for all $s \in S$, $t_{\alpha+1}(y_s) \neq t_\alpha(y_s)$ for just finitely many ordinals α (the same α's for all $s \in S$). Furthermore, $\bigvee_{s\in S} t_\alpha(y_s) = t_\alpha(1)$ for all α, by the above lemma. Thus, $t_{\alpha+1}(1) \neq t_\alpha(1)$ for finitely many ordinals α. Notice also that $t_\alpha(x) = t_\alpha(1) \wedge x$ for all α, thus $t_{\alpha+1}(x) \neq t_\alpha(x)$ implies $t_{\alpha+1}(1) \neq t_\alpha(1)$ and so, $t_{\alpha+1}(x) \neq t_\alpha(x)$ for finitely many ordinals α. Hence, $\mathrm{G.dim}([0,x]) = \sup\{\alpha+1 : t_{\alpha+1}(x) \neq t_\alpha(x)\} = \max\{\alpha+1 : t_{\alpha+1}(x) \neq t_\alpha(x)\}$ is a successor ordinal. ☐

Proposition 3.13. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal. Then,*

(1) $x \in [0, t_\alpha(1)]$ *if and only if* $\mathrm{G.dim}([0,x]) \leqslant \alpha$*;*

(2) *given a qframe $(L', \leqslant)$ and a homomorphism of qframes $\phi : L \to L'$, $\phi(t_\alpha(L)) \leqslant t_\alpha(L')$.*

Proof. (1) By part (3) of Lemma 3.7, $\mathrm{G.dim}([0, t_\alpha(1)]) \leqslant \alpha$ and so, by part (1) of the same lemma, $\mathrm{G.dim}([0,x]) \leqslant \mathrm{G.dim}([0, t_\alpha(1)]) \leqslant \alpha$ for all $x \in [0, t_\alpha(1)]$. On the other hand, if $\mathrm{G.dim}([0,x]) \leqslant \alpha$, then $x \leqslant t_\alpha(1)$ by construction.
(2) is an application of part (1) and Lemma 3.9. ☐

By part (2) of the above proposition we can give the following:

Definition 3.14. *Let α be an ordinal. Given a qframe $(L, \leqslant)$, we let $T_\alpha(L) = [0, t_\alpha(1)]$, while, given a homomorphism of qframes $\phi : L \to L'$, we let $T_\alpha(\phi) : T_\alpha(L) \to T_\alpha(L')$ be the restriction of ϕ. This defines a covariant functor $T_\alpha : \mathrm{QFrame} \to \mathrm{QFrame}$ that we call α-*torsion functor.

It is not difficult to show that T_α is compatible with the composition of morphisms, so that the above definition is correct. Notice that the α-torsion functor is an idempotent and hereditary preradical in the terminology of [2].

In the rest of this section we study the following equivalence relation induced by Gabriel dimension:

Definition 3.15. *Let $(L, \leqslant)$ be a qframe, let α be an ordinal and define the following relation between two elements x and y in L:*

$$(x, y) \in \mathcal{R}_\alpha \quad \textit{if and only if} \quad (\mathrm{G.dim}([x \wedge y, x \vee y]) \leqslant \alpha)\,.$$

We also use the notation $x \sim_\alpha y$ to say that $(x, y) \in \mathcal{R}_\alpha$.

Lemma 3.16. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal, then $\mathcal{R}_\alpha$ is a strong congruence on L.*

Proof. The fact that $\mathcal{R}_\alpha$ is a congruence follows by Lemma 3.7 (2) and [3, Proposition 2.4]. Furthermore, given $x \in L$, let us show that $\bigvee[x] \in [x]$. In fact,

$$\mathrm{G.dim}\left(\left[x, \bigvee_{y\in[x]} y\right]\right) = \mathrm{G.dim}\left(\left[x, \bigvee_{y\in[x]} x \vee y\right]\right) = \sup\{\mathrm{G.dim}([x, x\vee y]) : y \in [x]\} \leqslant \alpha\,,$$

by Lemma 3.7 (3). Thus, $\mathcal{R}_\alpha$ is a strong congruence. ☐

We denote by $Q_\alpha(L)$ the quotient of L over $\mathcal{R}_\alpha$ and by $\pi_\alpha : L \to Q_\alpha(L)$ the canonical surjective homomorphism.

Proposition 3.17. *Let $(L, \leqslant)$ and $(L', \leqslant)$ be qframes, let $\phi : L \to L'$ be a homomorphism of qframes, and let α be an ordinal.*

(1) *If $x \sim_\alpha y$ in L, then $\phi(x) \sim_\alpha \phi(y)$ in L';*

(2) $\mathrm{G.dim}(Q_\alpha(T_{\alpha+1}(L))) \leqslant 1$, *that is, $Q_\alpha(T_{\alpha+1}(L))$ is semi-Artinian for any ordinal α.*

Proof. (1) By Lemma 3.9, $\mathrm{G.dim}([x \wedge y, x \vee y]) \geqslant \mathrm{G.dim}(\phi([x \wedge y, x \vee y])) = \mathrm{G.dim}([\phi(x \wedge y), \phi(x) \vee \phi(y)])$. Furthermore, $\phi(x) \wedge \phi(y) \geqslant \phi(x \wedge y)$ and so $\mathrm{G.dim}([\phi(x) \wedge \phi(y), \phi(x) \vee \phi(y)]) \leqslant \mathrm{G.dim}[\phi(x \wedge y), \phi(x) \vee \phi(y)] \leqslant \alpha$, by Lemma 3.7 (1).
(2) Let $S = [a, b]$ be an α-simple segment of $T_{\alpha+1}(L)$. Then, $\pi_\alpha(S)$ is an atom since a is not α-equivalent to b (as $\mathrm{G.dim}(S) = \alpha + 1$) and b is α-equivalent to any $c \in (a, b]$ (as $\mathrm{G.dim}([c, b]) \leqslant \alpha$). If $Q_\alpha(T_{\alpha+1}(L)) = 0$ there is nothing to prove, otherwise choose an element $x \in T_{\alpha+1}(L)$ such that $\pi_\alpha(t_{\alpha+1}(1)) \neq \pi_\alpha(x) \in Q_\alpha(T_{\alpha+1}(L))$ and let $\bar{x} = \bigvee[x] \in T_{\alpha+1}(L)$. Notice that $\mathrm{G.dim}([\bar{x}, t_{\alpha+1}(1)]) = \alpha + 1$ (otherwise $[x] = [t_{\alpha+1}(1)]$). By definition of Gabriel dimension, there exists $\bar{y} \in T_{\alpha+1}(L)$ such that $[\bar{x}, \bar{y}]$ is β-simple for some $\beta < \alpha + 1$ and, since $\bar{y} \notin [\bar{x}]$, we have $\beta = \alpha$. By the previous discussion, $[\pi_\alpha(x), \pi_\alpha(\bar{y})]$ is 0-simple. One can conclude by Corollary 3.8. □

By part (1) of the above proposition, we can give the following:

Definition 3.18. *Let α be an ordinal. Given a qframe $(L, \leqslant)$, we let $Q_\alpha(L) = L_{/\mathcal{R}_\alpha}$, while, given a homomorphism of qframes $\phi : L \to L'$, we let $Q_\alpha(\phi) : Q_\alpha(L) \to Q_\alpha(L')$ be the induced homomorphism. This defines a functor $Q_\alpha : \mathrm{QFrame} \to \mathrm{QFrame}$ that we call* α-localization functor.

It is not difficult to show that Q_α is compatible with the composition of morphisms, so that the above definition is correct.

4 Main Theorems

Let V be a nonempty finite set and denote by S_V the symmetric group on V. Given two permutations σ_1 and $\sigma_2 \in S_V$ we let

$$d_V(\sigma_1, \sigma_2) = \frac{|\{v \in V : \sigma_1(v) \neq \sigma_2(v)\}|}{|V|},$$

be the *normalized Hamming distance* between σ_1 and σ_2.

Definition 4.1. *Let G be a group, let $K \subseteq G$ be a subset and let $\varepsilon > 0$. Given a finite set V, a* (K, ε)-quasi-action *of G on V is a map $\varphi : G \to S_V$ such that:*

(QA.1) *$\varphi(e) = id_V$;*

(QA.2) *for all k_1 and $k_2 \in K$, $d_V(\varphi(k_1 k_2), \varphi(k_1)\varphi(k_2)) \leqslant \varepsilon$;*

(QA.3) *for all $k_1 \neq k_2 \in K$, $d_V(\varphi(k_1), \varphi(k_2)) \geqslant 1 - \varepsilon$.*

Whenever we have a quasi-action we adopt the following notation. Given two subsets $V' \subseteq V$ and $G' \subseteq G$, we let $G'V' = \{\varphi_g(v) : g \in G',\ v \in V'\}$. In case $V' = \{v\}$ is a singleton set we let $G'v = G'\{v\}$. Similarly, if $G' = \{g\}$ is a singleton, $gV' = \{g\}V'$. Furthermore, $gv = \varphi_g(v)$ for all $v \in V$ and $g \in G'$.

For finitely generated groups, the following definition of sofic group is equivalent to the definition given in [30] and [20] (see [9]).

Definition 4.2. *A group G is sofic if, for any subset $K \subseteq G$ and for any positive constant ε, there exists a finite set V and a (K, ε)-quasi-action of G on V.*

4.1 The 1-dimensional case

I've learnt the arguments used in the proof of the following lemma while reading [17, proof of Proposition 4.4] and [30, proof of Lemma 3.1]. Also Lemma 4.4 is inspired to the argument used by Weiss to show surjunctivity of sofic groups.

Lemma 4.3. *Let G be a group, let K be a finite symmetric subset of G and let $H = KK$. Choose $n \in \mathbb{N}_{\geqslant 2}$, let ε be a positive constant such that $\varepsilon < \frac{1}{2n|H|^2}$, let V be a finite set, let $\varphi : G \to S_V$ be an (H, ε)-quasi-action of G on V and define the following set:*

$$\bar{V} = \{v \in V : hv \neq h'v \text{ and } (h_1h_2)v = h_1(h_2v), \text{ for all } h \neq h' \in H,\ h_1, h_2 \in H\}.$$

Then, the following statements hold true:

(1) *$|\bar{V}| \geqslant (1 - 1/n)|V|$;*

(2) *there is a subset $W \subseteq \bar{V}$ such that $Kv \cap Kw = \varnothing$ for all $v \neq w \in W$ and $|W| \geqslant |V|/2|H|$.*

Proof. (1) A given $v \in V$ belongs to $\bar{V}$ if and only if it satisfies the following two conditions:

(a) $\varphi_{h_1}(v) \neq \varphi_{h_2}(v)$ for all $h_1 \neq h_2 \in H$;

(b) $\varphi_{h_1h_2}(v) = \varphi_{h_1}(\varphi_{h_2}(v))$ for all $h_1, h_2 \in H$.

There are less than $|H|^2$ equations in (a) and each of these equations can fail for at most $\varepsilon|V|$ elements v in V. Similarly, there are $|H|^2$ equations in (b) and each of these equations can fail for at most $\varepsilon|V|$ elements $v \in V$. Thus, the cardinality of $\bar{V}$ is at least

$$|V| - (|H|^2\varepsilon|V| + |H|^2\varepsilon|V|) \geqslant |V|(1 - 2|H|^2\varepsilon) \geqslant |V|(1 - 1/n).$$

(2) Let W be a maximal subset of $\bar{V}$ with the property that $Kv \cap Kw = \varnothing$ for all $v \neq w \in W$. We claim that HW contains $\bar{V}$. In fact, if there is $v \in \bar{V}$ such that $v \notin HW$, this means that, for all $w \in W$, $Kv \cap Kw = \varnothing$, contradicting the maximality of W. Thus, $|\bar{V}| \leqslant |WH| \leqslant |W||H|$. To conclude, use that $2|\bar{V}| \geqslant |V|$ by part (1) and the choice of n. ☐

Lemma 4.4. *In the same setting of Lemma 4.3, let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be two qframes of finite length and consider a homomorphism of qframes $\Phi : L_1 \to L_2$. Let $l \in \mathbb{N}_{\geqslant 1}$ and suppose that*

(1) *there is distinguished family of elements $\{\bar{x}_v : v \in K\bar{V}\}$ such that*

 (1.1) *$\bigvee_{K\bar{V}} \bar{x}_v = 1$;*

 (1.2) *$\ell(\bar{x}_v) = l$, for all $v \in K\bar{V}$;*

(2) *$\ell\left(\bigvee_{v \in Kw} \Phi(\bar{x}_v)\right) \leqslant |K|l - 1$, for all $w \in \bar{V}$.*

Then, $\ell(\mathrm{Im}(\Phi)) \leqslant \left(1 - \frac{1}{2|H|l}\right)|V|l$.

Proof. Choose a $W \subseteq \bar{V}$ as in part (2) of Lemma 4.3. By Lemma 2.17,

$$\ell(\Phi(L_1)) = \ell\left(\bigvee_{v \in K\bar{V}} \Phi(\bar{x}_v)\right) \leqslant \ell\left(\bigvee_{v \in K\bar{V} \setminus KW} \Phi(\bar{x}_v)\right) + \ell\left(\bigvee_{v \in KW} \Phi(\bar{x}_v)\right).$$

Furthermore,

$$\ell\left(\bigvee_{v\in KW}\Phi(\bar{x}_v)\right)\leqslant\sum_{w\in W}\ell\left(\bigvee_{v\in Kw}\Phi(\bar{x}_v)\right)\leqslant|W|(|K|l-1)\,.$$

By the choice of W, $|K\bar{V}\backslash KW|=|K\bar{V}|-\sum_{w\in W}|Kw|=|K\bar{V}|-|W||K|$ and, by Lemma 2.18, $\ell\left(\bigvee_{v\in K\bar{V}\backslash KW}\Phi(\bar{x}_v)\right)\leqslant\ell\left(\bigvee_{v\in K\bar{V}\backslash KW}\bar{x}_v\right)$, thus

$$\ell\left(\bigvee_{v\in K\bar{V}\backslash KW}\bar{x}_v\right)\leqslant\sum_{v\in K\bar{V}\backslash KW}\ell(\bar{x}_v)=|K\bar{V}\backslash KW|l=(|K\bar{V}|-|W||K|)l\leqslant(|V|-|W||K|)l\,,$$

Putting together all these data, we get

$$\ell(\Phi(L_1))\leqslant|W|(|K|l-1)+(|V|-|W||K|)l=-|W|+|V|l\leqslant\left(1-\frac{1}{2|H|l}\right)|V|l\,.$$

□

Theorem 4.5. *Let M be a qframe, let G be a sofic group, let $\rho:G\to\mathrm{Aut}(M)$ be a right action of G on M (we let $\rho(g)=\rho_g$ for all $g\in G$) and let $\phi:M\to M$ be a G-equivariant homomorphism of qframes, that is, $\rho_g\phi=\phi\rho_g$, for all $g\in G$. Choose an element $\bar{y}\in M$ such that*

(a) $\ell(\bar{y})=l<\infty$;

(b) *the family $\{\bar{y}_g:g\in G\}$ is join-independent, where $\bar{y}_g=\rho_g(\bar{y})$ for all $g\in G$;*

(c) *there exists a finite symmetric subset $F\subseteq G$ such that $\phi(\bar{y})\leqslant\bigvee_{g\in F}\bar{y}_g$ and $e\in F$.*

Fix an F as in (c) and let K be a finite symmetric subset of G containing F. Then, the following conditions are mutually exclusive:

(1) $\bar{y}\leqslant\bigvee_{g\in K}\phi(\bar{y}_g)$;

(2) $\ell\left(\bigvee_{g\in K}\phi(\bar{y}_g)\right)\leqslant|K|l-1$.

Proof. Assume, looking for a contradiction, that both (1) and (2) are verified. We start by constructing some objects to which we want to apply Lemmas 4.3 and 4.4.

First we construct the objects mentioned in Lemma 4.3. Choose a positive integer $n\geqslant 2|H|l$, let $H=KK$, let ε be a positive constant such that $\varepsilon<\frac{1}{2n|H|^2}$, let V be a finite set, let $\varphi:G\to S_V$ be an (H,ε)-quasi-action of G on V and define

$$\bar{V}=\{v\in V:hv\neq h'v\text{ and }(h_1h_2)v=h_1(h_2v),\ \text{for all }h\neq h'\in H,\,h_1,h_2\in H\}\,.$$

Let us construct now the objects mentioned in Lemma 4.4. For a subset $G'\subseteq G$, we use the notation $\bar{y}_{G'}=\bigvee_{g\in G'}\bar{y}_g$ and, for all $v\in V$, we let $Q_v^{G'}$ be a qframe isomorphic to $[0,\bar{y}_{G'}]$. For all $v\in V$, we identify both $Q_v^e=Q_v^{\{e\}}$ and Q_v^K with sub-qframes of Q_v^H in such a way that there is an isomorphism of qframes

$$q_v:Q_v^H\longrightarrow[0,\bar{y}_H]\,,$$

such that $q_v(Q_v^e)=[0,\bar{y}]$ and $q_v(Q_v^K)=[0,\bar{y}_K]$. For all $v\in\bar{V}$, the map $\sigma_v:Hv\to H$ such that $\sigma_v(hv)=h$ is well defined and bijective. So, given $v\in\bar{V}$ and $w\in Hv$ we let

$$q_v^w:Q_w^H\xrightarrow{\sim}[0,\bar{y}_{H\sigma_v(w)}]$$

be the composition $q_v^w = \rho_{\sigma_v(w)} q_w$. Let us introduce the following notation for all $G' \subseteq G$:

$$Q^{G'} = \prod_{v \in \bar{V}} Q_v^{G'}, \quad \forall G' \subseteq G.$$

For all $v \in \bar{V}$, we denote by $\iota_v^{G'} : Q_v^{G'} \to Q^{G'}$ the canonical inclusion in the product. Consider, for all $v \in \bar{V}$, the following homomorphism of qframes:

$$\Psi_v : Q^H \longrightarrow [0, \bar{y}_{HH}] \text{ such that } (a_w)_{w \in \bar{V}} \mapsto \bigvee_{w \in Hv \cap \bar{V}} q_v^w(a_w).$$

We define a relation $\mathcal{R} \subseteq Q^H \times Q^H$ as follows:

$$(a, b) \in \mathcal{R} \iff \Psi_v(a) = \Psi_v(b) \ \forall v \in \bar{V}.$$

This defines a strong congruence on Q^H and, by restriction, on Q^K. Let $L_1 = Q^K/\mathcal{R}$ and $L_2 = Q^H/\mathcal{R}$ and let $\pi_1 : Q^K \to L_1$ and $\pi_2 : Q^H \to L_2$ be the canonical projections. For all $v \in \bar{V}$, let $\Phi_v : Q_v^K \to Q_v^H$ be the unique map such that $q_v\Phi_v(x) = \phi(q_v(x))$, for all $x \in Q_v^K$. We let $\Phi : Q^K \to Q^H$ be the product of these maps, that is, $\Phi(x_v)_{v \in \bar{V}} = (\Phi_v(x_v))_{v \in \bar{V}}$. Given two elements $a \sim b \in Q^K$, $\Phi(a) \sim \Phi(b)$. In fact, for all $v \in \bar{V}$,

$$\begin{aligned}
\Psi_v\Phi(a) &= \bigvee_{w \in Hv \cap \bar{V}} \rho_{\sigma_v(w)} q_w \Phi_w(a_w) = \bigvee_{w \in Hv \cap \bar{V}} \rho_{\sigma_v(w)} \phi q_w(a_w) = \phi\left(\bigvee_{w \in Hv \cap \bar{V}} q_v^w(a_w)\right) \\
&= \phi\left(\bigvee_{w \in Hv \cap \bar{V}} q_v^w(b_w)\right) = \ldots = \Psi_v\Phi(b).
\end{aligned}$$

Let $\bar{\Phi} : L_1 \to L_2$ be the unique map such that $\bar{\Phi}\pi_1 = \pi_2\Phi$. One verifies that $\bar{\Phi}$ is a morphism of qframes.

Now that the setting is constructed we need to verify that the hypotheses (1) and (2) of Lemma 4.4 are satisfied. For all $v \in \bar{V}$ and $k \in K$ we let $x_k^v = \iota_v^K(q_v^{-1}(\bar{y}_k))$. Let us show that $x_k^v \sim x_{k'}^{v'}$ if and only if $kv = kv'$. Indeed, given v, $v' \in \bar{V}$ and k, $k' \in K$ such that $x_k^v \sim x_{k'}^{v'}$, notice that

$$\Psi_v(x_k^v) = \bar{y}_k \quad \text{and} \quad \Psi_v(x_{k'}^{v'}) = \rho_{\sigma_v(v')} \bar{y}_{k'}$$

if $v' \in Hv$, otherwise it is 0. Thus, $\sigma_v(v')k' = k$, that is, $v' = \sigma_v(v')v = (k')^{-1}kv$, so $k'v' = kv$ (here we are using that v, $v' \in \bar{V}$). Hence, given $w = kv \in K\bar{V}$, we can define $\bar{x}_w = \pi_1(x_k^v)$ without any ambiguity. Clearly $\bigvee_{v \in K\bar{V}} \bar{x}_v = 1$, let us show that the family $\{\bar{x}_v : v \in K\bar{V}\} \subseteq L_1$ is join-independent. Indeed, given $k'v' \in K\bar{V}$,

$$\bar{x}_{k'v'} \wedge \bigvee_{k'v' \neq kv \in K\bar{V}} \bar{x}_{kv} = \pi_1\left(x_{k'}^{v'} \wedge \bigvee_{k'v' \neq kw \in K\bar{V}} x_k^v\right) = \pi_1(0) = 0,$$

where the first equality comes from the definition of the $\bar{x}_w$ and the properties of π_1 (see Lemma 2.9), while the second equality holds since the family $\{x_k^v : kv \in K\bar{V}\} \subseteq Q^K$ is join-independent.

Furthermore, for all $w \in \bar{V}$:

$$\begin{aligned}
\ell\left(\bigvee_{v \in Kw} \bar{\Phi}(\bar{x}_v)\right) &= \ell\left(\bigvee_{v \in Kw} \bar{\Phi}\pi_1(Q_v^K)\right) = \ell\left(\bigvee_{v \in Kw} \pi_2\Phi_v(Q_v^K)\right) \leqslant \ell\left(\bigvee_{v \in Kw} \Phi(Q_v^K)\right) \\
&\leqslant \ell(\phi([0, \bar{y}_K])) \leqslant |K|l - 1.
\end{aligned}$$

In the last part of the proof we obtain the contradiction we were looking for. Indeed, we claim that the restriction of π_2 to Q^e is injective and that $\pi_2(Q^e) \subseteq \bar{\Phi}(L_1)$. In fact, let $a = (a_v)_{v\in\bar{V}}$ and $b = (b_v)_{v\in\bar{V}} \in Q^e$ and suppose that $\pi_2(a) = \pi_2(b)$, that is, $a \sim b$. For all $v \in \bar{V}$ and $w \in Hv \cap \bar{V}$, by construction, $q_v^w(a_w)$, $q_v^w(b_w) \leqslant \bar{y}_{\sigma_v(w)}$. So, using modularity and the independence of the family $\{\bar{y}_g : g \in G\}$,

$$q_v(a_v) = q_v(a_v) \vee 0 = h_v(a_v) \vee \left(\bigvee_{v\neq w\in Hv\cap\bar{V}} q_v^w(a_w) \wedge \bar{y} \right) = \bar{y} \wedge \left(q_v(a_v) \vee \bigvee_{v\neq w\in Hv\cap\bar{V}} q_v^w(a_w) \right)$$

$$= \bar{y} \wedge \Psi_v(a) = \bar{y} \wedge \Psi_v(b) = \ldots = q_v(b_v)\,,$$

that is, $a_v = b_v$, for all $v \in \bar{V}$. Our second claim follows by construction and the hypothesis (1). Also recalling the estimate for $|\bar{V}|$ given in Lemma 4.3, the two claims we just verified imply that

$$\ell(\mathrm{Im}(\bar{\Phi})) \geqslant \ell(\pi_2(Q^{(e)})) = \ell(Q^{(e)}) = |\bar{V}|l \geqslant \left(1 - \frac{1}{n}\right)|V|l\,.$$

Furthermore, by Lemma 4.4, $\ell(\mathrm{Im}(\bar{\Phi})) < \left(1 - \frac{1}{2|H|l}\right)|V|l$. Thus, $n < 2|H|l$, which is a contradiction. □

4.2 Higher dimensions

Lemma 4.6. *Let $(M, \leqslant)$ be a qframe, let G be a group, let $\rho : G \to \mathrm{Aut}(M)$ be a right action of G on M and consider an algebraic G-equivariant homomorphism of qframes $\phi : M \to M$. Suppose that there exists an element $y \in M$ such that $[0, y]$ is finitely generated and such that, letting $y_g = \rho_g(y)$ for all $g \in G$, the family $\{y_g : g \in G\}$ is a basis for M. Then,*

(1) *ϕ is surjective if and only if there exists a finite subset $K \subseteq G$ such that $y \leqslant \bigvee_{g\in K} \phi(y_g)$;*

(2) *ϕ is not injective if and only if there exist a finite subset $K \subseteq G$ and $0 \neq x \leqslant \bigvee_{g\in K} y_g$ such that $\phi(x) = 0$.*

Proof. (1) Suppose that ϕ is surjective, then $\bigvee_{g\in G} \phi(y_g) = \phi(1) = 1$. By Lemma 2.6, one can find a finite subset $K \subseteq G$ such that $y \leqslant \bigvee_{g\in G} \phi(y_g)$. On the other hand, if there exists $K \subseteq G$ such that $y \leqslant \bigvee_{g\in K} \phi(y_g)$, then $y_h \leqslant \bigvee_{g\in Kh^{-1}} \phi(y_g) \leqslant \phi(1)$ for all $h \in G$. Thus, $1 = \bigvee_{h\in G} y_h \leqslant \phi(1)$ and so ϕ is surjective.

(2) By the algebraicity of ϕ, if ϕ is not injective, there is a non-trivial element $x' \in \mathrm{Ker}(\phi)$. By Lemma 2.6, there exists a finite subset $K \subseteq G$ such that $x' \wedge \bigvee_{g\in K} y_g \neq 0$, so that $x = x' \wedge \bigvee_{g\in K} y_g$ is the element we were looking for. The converse is trivial. □

Theorem 4.7. *Let $(M, \leqslant)$ be a qframe, let G be a sofic group, let $\rho : G \to \mathrm{Aut}(M)$ be a right action of G on M and consider a surjective algebraic G-equivariant homomorphism of qframes $\phi : M \to M$. For a given element $y \in M$ such that $[0, y]$ is compact, consider the following conditions:*

(a$_*$) *$[0, y]$ is Noetherian;*

(a$_*'$) K.dim$([0, y])$ *exists and there is a homomorphism of qframes $\psi : M \to M$ such that $\phi\psi = id$;*

(b$_*$) *letting $y_g = \rho_g(y)$ for all $g \in G$, the family $\{y_g : g \in G\}$ is a basis for M.*

If (b_*) *and either* (a_*) *or* (a'_*) *hold, then* ϕ *is injective.*

Proof. Suppose, looking for a contradiction, that ϕ is not injective. Suppose that (b_*) is verified, so by Lemma 4.6, there exists a finite subset K of G such that

(1_*) $y \leqslant \bigvee_{g\in K}\phi(y_g)$;

(2_*) there exists $0 \neq x \leqslant \bigvee_{g\in K} y_g$ such that $\phi(x) = 0$.

Furthermore, since $[0, y]$ is compact, also $[0, \phi(y)]$ is compact and so there exists a finite subset $F \subseteq G$ such that

(3_*) $\phi(y) \leqslant \bigvee_{g\in F} y_g$.

In case (a_*) is verified, by Lemma 3.12 there exists an ordinal α such that $\mathrm{G.dim}([0, \mathrm{Ker}(\phi)]) = \alpha + 1$. On the other hand, if (a'$_*$) is verified, we let α be any ordinal such that $t_\alpha(x) \neq t_{\alpha+1}(x)$. In both cases, let $\bar M = Q_\alpha(T_{\alpha+1}(M))$ and denote by $\pi : T_{\alpha+1}(M) \to \bar M$ the canonical projection. We let $\bar x = \pi(t_{\alpha+1}(x))$ and $\bar y = \pi(t_{\alpha+1}(y))$. There is an induced right action of G on $\bar M$, $\bar\rho : G \to \mathrm{Aut}(\bar M)$, where $\bar\rho_g = Q_\alpha(T_{\alpha+1}(\rho_g))$ for all $g \in G$. Of course, the map $\bar\phi = Q_\alpha(T_{\alpha+1}(\phi)) : \bar M \to \bar M$ is G-equivariant. One can prove that $\bar\rho_g(\bar y) = \pi(t_{\alpha+1}(y_g))$, for all $g \in G$, and so, whenever (b_*) is verified, the family $\{\bar y_g : g \in G\}$, where $\bar y_g = \bar\rho_g(\bar y)$, is a basis of $\bar M$ (it is clear that $\bigvee \bar y_g = 1$, to see that this family is join-independent use that the canonical projection commutes with joins and finite meets by Lemma 2.9).

Suppose that (a_*) is verified. By Proposition 3.17 (2), $[0, \bar y]$ is semi-Artinian and, by (a_*), it is also Noetherian. Thus, $\ell(\bar y) = l < \infty$. Notice that, by ($3_*$), $\bar\phi(\bar y) \leqslant \bigvee_{g\in F}\bar y_g$ and, by (1_*), $t_{\alpha+1}(y) \in [0, \bigvee_{g\in K}\phi(y_g)]$, thus there exists $z \leqslant \bigvee_{g\in K} y_g$ such that $\phi(z) = t_{\alpha+1}(y)$. By the algebraicity of ϕ and Lemma 3.7 (2), $\mathrm{G.dim}([0, z]) = \max\{\mathrm{G.dim}([0, \mathrm{Ker}(\phi) \wedge z]), \mathrm{G.dim}([0, t_{\alpha+1}(y)])\} = \alpha + 1$, thus $z \in [0, \bigvee_{g\in K} t_{\alpha+1}(y_g)]$. Applying π, we obtain an element $\pi(z) \in [0, \bigvee_{g\in K}\bar y_g]$ such that $\bar\phi(\pi(z)) = \pi(\phi(z)) = \bar y$. Thus, $\bar y \leqslant \bigvee_{g\in K}\bar\phi(\bar y_g)$. By the choice of α, $\mathrm{Ker}(\bar\phi) \neq 0$ and so, by Lemma 2.6, there exists a finite subset $F' \subseteq G$ such that $\mathrm{Ker}(\bar\phi) \wedge \bigvee_{g\in F'}\bar y_g \neq 0$. Let K' be a finite symmetric subset of G which contains both F' and K, then

$$\bar y \leqslant \bigvee_{g\in K'}\bar\phi(\bar y_g) \quad \text{and} \quad \ell\left(\bigvee_{g\in K'}\bar\phi(\bar y_g)\right) \leqslant |K'|l - 1\,,$$

by the above discussion and Lemma 2.20. These two conditions cannot happen for the same K' by Theorem 4.5, so we get a contradiction.

Suppose now that (a'_*) is verified. We define $\bar\psi = Q_\alpha(T_{\alpha+1}(\psi)) : \bar M \to \bar M$, so that $\bar\phi\bar\psi = id$. Consider the socle $\mathrm{Soc}(\bar M) = [0, s(\bar M)]$ and notice that, since $\{\bar y_g : g \in G\}$ is join-independent, $s(\bar M) = \bigvee_{g\in G} s([0, \bar y_g])$. Since $[0, \bar y]$ is semi-Artinian and it has Krull dimension, then it is Artinian, thus, it has a socle of finite length: let $l = \ell(s(\bar y))$. By the choice of α, $\bar x \neq 0$ and $\bar x \wedge s(\bar M) \neq 0$, since, being $\bar M$ semi-Artinian, $s(\bar M)$ is essential in $\bar M$. Since $\mathrm{Soc}(\bar M)$ is fully invariant (see Lemma 2.22 (4)), $\bar\phi\restriction_{\mathrm{Soc}(\bar M)}\bar\psi\restriction_{\mathrm{Soc}(\bar M)} = id_{\mathrm{Soc}(\bar M)}$. The family $\{s(\bar y_g) : g \in G\}$ is clearly join-independent. Furthermore, using the fact that $[0, s(\bar y)]$ is compact (since it has finite length), also $[0, \bar\phi(s(\bar y))]$ and $[0, \bar\psi(s(\bar y))]$ are compact, so there exists a finite subset $F' \subseteq G$ such that $\bar\phi(s(\bar y))$, $\bar\psi(s(\bar y)) \leqslant \bigvee_{g\in F'} s(\bar y_g)$. Let $K' \subseteq G$ be a finite symmetric subset that contains both F' and K, then

$$s(\bar y) = \bar\phi(\bar\psi(s(\bar y))) \leqslant \bar\phi\left(\bigvee_{g\in K'} s(\bar y_g)\right) \leqslant \bigvee_{g\in K'}\bar\phi(s(\bar y_g)) \quad \text{and} \quad \ell\left(\bigvee_{g\in K'}\bar\phi(s(\bar y_g))\right) \leqslant |K'|l - 1\,,$$

by Lemma 2.20 and the fact that $\bar\phi(\bar x \wedge s(\bar M)) = 0$. This is a contradiction by Theorem 4.5. $\square$

5 Applications

5.1 Stable Finiteness of crossed products

Given a group G and a ring R, a *crossed product* $R{*}G$ of R with G is a ring constructed as follows: as a set, $R{*}G$ is the collection of all the formal sums

$$\sum_{g\in G} r_g \underline{g}\,,$$

with $r_g \in R$ and $r_g = 0$ for all but finite $g \in G$, and where any $\underline{g}$ is a symbol uniquely assigned to $g \in G$. The sum in $R{*}G$ is defined component-wise exploiting the addition in R:

$$\left(\sum_{g\in G} r_g\underline{g}\right) + \left(\sum_{g\in G} s_g\underline{g}\right) = \sum_{g\in G}(r_g + s_g)\underline{g}\,.$$

In order to define a product in $R{*}G$, we need to specify two maps

$$\sigma : G \to \mathrm{Aut}_{ring}(R) \quad \text{and} \quad \tau : G \times G \to U(R)\,,$$

where $\mathrm{Aut}_{ring}(R)$ is the group of ring automorphisms of R and $U(R)$ is the group of units of R. Given $g \in G$ and $r \in R$ we denote the image of r via the automorphism $\sigma(g)$ by $r^{\sigma(g)}$. We suppose also that σ and τ satisfy the following conditions for all $r \in R$ and g, g_1, g_2 and $g_3 \in G$:

(Cross.1) $\sigma(e) = 1$ and $\tau(g, e) = \tau(e, g) = 1$;

(Cross.2) $\tau(g_1, g_2)\tau(g_1g_2, g_3) = \tau(g_2, g_3)^{\sigma(g_1)}\tau(g_1, g_2g_3)$;

(Cross.3) $r^{\sigma(g_2)\sigma(g_1)} = \tau(g_1, g_2)r^{\sigma(g_1g_2)}\tau(g_1, g_2)^{-1}$.

The product in $R{*}G$ is defined by the rule $(r\underline{g})(s\underline{h}) = rs^{\sigma(g)}\tau(g, h)\underline{gh}$, together with bilinearity, that is

$$\left(\sum_{g\in G} r_g\underline{g}\right)\left(\sum_{g\in G} s_g\underline{g}\right) = \sum_{g\in G}\left(\sum_{h_1h_2=g} r_{h_1}s_{h_2}^{\sigma(h_1)}\tau(h_1, h_2)\right)\underline{g}\,.$$

$R{*}G$ is an associative and unitary ring. Of course the definition of $R{*}G$ does not depend only on R and G, but also on σ and τ. Anyway we prefer the compact (though imprecise) notation $R{*}G$ to something like $R[G, \rho, \sigma]$. The easiest example of crossed group ring is the group ring $R[G]$, which corresponds to trivial maps σ and τ. For more details on this kind of construction we refer to [27].

Lemma 5.1. *Let R be a ring, let M_R and N_R be right R-modules and let $\phi : M \to N$ be a homomorphism of right R-modules. Then, $(\mathcal{L}(M), \leqslant)$ and $(\mathcal{L}(N), \leqslant)$ are qframes and the map*

$$\Phi : \mathcal{L}(M) \to \mathcal{L}(N) \quad \textit{such that} \quad \Phi(K) = \phi(K)$$

is a homomorphism of qframes. Furthermore, if ϕ is surjective, then Φ is surjective and algebraic, and, in this case, Φ is injective if and only if ϕ is injective.

Proof. It is well-known that the family of submodules with the usual ordering is a qframe (the maximum of $\mathcal{L}(M)$ is M, while its minimum is 0, furthermore, join and meet are given by sum and intersection respectively). Furthermore, it is easy to verify that Φ is a semi-lattice homomorphism

which commutes with arbitrary joins. To show that Φ sends segments to segments, let $K_1 \leqslant K_2 \in \mathcal{L}(M)$ and consider $K \in [\Phi(K_1), \Phi(K_2)]$. Then,

$$\begin{aligned} K &= \Phi(\phi^{-1}K) = \Phi(\phi^{-1}K \cap \phi^{-1}\phi(K_2)) \\ &= \Phi(\phi^{-1}K \cap (K_2 + \mathrm{Ker}(\phi))) = \Phi((\phi^{-1}K \cap K_2) + \mathrm{Ker}(\phi)) \\ &= \Phi(\phi^{-1}K \cap K_2) + \Phi(\mathrm{Ker}(\phi)) = \Phi(\phi^{-1}K \cap K_2)\,, \end{aligned}$$

where in the first line we used that K is contained in the image of ϕ, while in the second line we used the modularity of $\mathcal{L}(M)$. Since $\phi^{-1}(K) \cap K_2 \in [K_1, K_2]$ we proved that Φ sends segments to segments, thus it is a morphism of qframes.

Suppose now that ϕ is surjective. Then, Φ is surjective as $\Phi(1) = \phi(M) = N$, which is the maximum of $\mathcal{L}(N)$. To show that Φ is algebraic, notice that $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\phi)$ and that, given K_1, $K_2 \in [\mathrm{Ker}(\phi), 1]$ such that $\Phi(K_1) = \Phi(K_2)$, we get

$$K_1 = K_1 + \mathrm{Ker}(\phi) = \phi^{-1}(\phi(K_1)) = \phi^{-1}(\phi(K_2)) = K_2 + \mathrm{Ker}(\phi) = K_2\,.$$

Finally, notice that ϕ is injective if and only if $\mathrm{Ker}(\phi) = \mathrm{Ker}(\Phi) = 0$, which happen, by the algebraicity of Φ, if and only if Φ is injective. $\square$

Given a crossed product $R{*}G$, there is a canonical injective ring homomorphism $R \to R{*}G$ such that $r \mapsto r\underline{e}$, which allows to identify R with a subring of $R{*}G$ and to consider $R{*}G$-modules also as R-modules in a natural way. On the other hand, there is no natural map $G \to R{*}G$ which is compatible with the operations, anyway the obvious assignment $g \mapsto \underline{g}$ respects the operations "modulo some units of R". This is enough to obtain a canonical right action of G on $\mathcal{L}_R(M)$, for any right R-module M:

Lemma 5.2. *Let R be a ring, let G be a group, fix a crossed product $R{*}G$, let M_{R*G} be a right $R{*}G$-module and let $\phi : M_{R*G} \to M_{R*G}$ be an endomorphism of right $R{*}G$-modules. Letting $\mathcal{L}_R(M)$ denote the qframe of R-submodules of M, the following map*

$$\rho : G \to \mathrm{Aut}(\mathcal{L}_R(M)) \quad \rho \mapsto \rho_g : \mathcal{L}_R(M) \to \mathcal{L}_R(M)\,,$$

where $\rho_g(K) = K\underline{g}$, for all $g \in G$ and $K \in \mathcal{L}_R(M)$ is a group anti-homomorphism. Furthermore, the endomorphism of qframes

$$\Phi : \mathcal{L}_R(M) \to \mathcal{L}_R(M) \quad \text{such that} \quad \Phi(K) = \phi(K)$$

is G-equivariant.

Proof. Let $N \in \mathcal{L}_R(M)$, $r \in R$ and $g \in G$. Then, $\rho_g(N)r = N\underline{g}r = Nr^{\sigma(g)}\underline{g} \subseteq N\underline{g}$ and so $\rho_g(N) \in \mathcal{L}_R(M)$. Let now $\{N_i : i \in I\}$ a family of elements in $\mathcal{L}_R(M)$, then

$$\rho_g\left(\sum_{i\in I} N_i\right) = \left(\sum_{i\in I} N_i\right)\overline{g} = \sum_{i\in I}(N_i\overline{g}) = \sum_{i\in I}\rho_g(N_i)$$

so ρ_g is a semi-lattice homomorphism which commutes with arbitrary joins. Furthermore, given g, $h \in G$ and $N \in \mathcal{L}_R(M)$,

$$\rho_g(\rho_h(N)) = \rho_g(N\underline{h}) = N\underline{h}\underline{g} = N\tau(h,g)\underline{hg} = N\underline{hg} = \rho_{hg}(N)\,,$$

where the fourth equality holds since $\tau(h,g) \in U(R)$. In particular, $\rho_g\rho_{g^{-1}} = \rho_{g^{-1}}\rho_g = id_{\mathcal{L}_R(M)}$ so, given a segment $[N_1, N_2]$ in $\mathcal{L}_R(M)$ and $N \in [\rho_g(N_1), \rho_g(N_2)]$, then $N = \rho_g(\rho_{g^{-1}}N)$ and

$\rho_{g^{-1}}N \in [N_1, N_2]$. Thus we proved that each ρ_g is a homomorphism of qframes and that ρ is a group anti-homomorphism.

Finally, let us show that $\rho_g\Phi = \Phi\rho_g$. Indeed, given $N \in \mathcal{L}_R(M)$,

$$\rho_g\Phi(N) = \phi(N)\underline{g} = \phi(N\underline{g}) = \Phi(\rho_g(N)),$$

where the third equality holds since ϕ is a homomorphism of left $R{*}G$-modules. □

Theorem 5.3. *Let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$, let N_R be a finitely generated right R-module and let $M = N \otimes R{*}G$. Then,*

(1) *if N_R is Noetherian, then any surjective $R{*}G$-linear endomorphism of M is injective;*

(2) *if N_R has Krull dimension, then $\mathrm{End}_{R*G}(M)$ is stably finite.*

Proof. The proof is an application of Theorem 4.7 and consists in translating the statement in a problem about qframes using the above lemmas.

Suppose first (1) and let $\phi : M \to M$ be a surjective endomorphism of right $R{*}G$-modules. Consider the qframe $\mathcal{L}_R(M)$ of all the right R-submodules of M (which is described in Lemma 5.1), with the right G-action described in Lemma 5.2. By the same lemma, ϕ induces a G-equivariant surjective algebraic homomorphism of qframes $\Phi : \mathcal{L}_R(M) \to \mathcal{L}_R(M)$.

Let $y = N \otimes \underline{e} \in \mathcal{L}_R(M)$, and notice that conditions (a_*) and (b_*) in Theorem 4.7 are verified for this choice of y. Thus, by the theorem, Φ is injective and this is equivalent to say that ϕ is injective by Lemma 5.1.

The proof of part (2) is analogous. □

The above theorem can be used to verify the conjecture for classes of groups that are not known to be sofic. Remember that, given two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of groups, a group is said to be $\mathcal{C}_1$-by-$\mathcal{C}_2$ provided there exists a short exact sequence

$$1 \to C_1 \to G \to C_2 \to 1$$

with $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$.

Corollary 5.4. *Let R be a right Noetherian ring an let G be a (finite-by-polycyclic)-by-sofic group. Then any crossed product $R{*}G$ is stably finite.*

Proof. Consider a short exact sequence $1 \to C_1 \to G \to C_2 \to 1$ with C_1 a finite-by-polycyclic group and C_2 sofic. Then, $R{*}G \cong (R{*}C_1){*}C_2$ for suitable choices of the crossed products on the right. It is well-known (see, for example, [7, Proposition 2.5]) that $R{*}C_1$ is a right Noetherian ring, thus $(R{*}C_1){*}C_2$ is stably finite by Theorem 5.3. □

Corollary 5.5. *Let R be a division ring and let G be a free-by-sofic group. Then $R[G]$ is stably finite.*

Proof. Consider a short exact sequence $1 \to C_1 \to G \to C_2 \to 1$ with C_1 a free group and C_2 sofic. It is known (see [14, Theorem 5.3.9]) that $R[C_1]$ embeds in a division ring D and so, $R[G] \cong R[C_1]{*}C_2$ embeds in $D{*}C_2$. One concludes applying Theorem 5.3. □

In the same line of the above corollaries, Federico Berlai [8] will use Theorem 5.3 to provide examples of groups that are not known to be sofic but that satisfy the Stable Finiteness Conjecture.

Let us conclude this subsection with an open question:

Question 5.6. *Let D be a division ring and let G be a group. Is it possible to find a sofic group H and a suitable crossed product such that $D{*}H \cong D[G]$? If this is not possible, can we choose $D{*}H$ (with H sofic) to be Morita equivalent to $D[G]$?*

Given a stably finite ring R, any subring of $\mathrm{Mat}_n(R)$, for any $n \in \mathbb{N}_+$, is stably finite, thus a positive answer to the above question would solve the Kaplansky Stable Finiteness Conjecture for all groups.

5.2 L-Surjunctivity

Let G be a group and let A be a set. The set of *configurations* over G in the alphabet A is the cartesian product $A^G = \{x : G \to A\}$. The left action of G on A^G defined by

$$gx(h) = x(g^{-1}h) \text{ for all } g, h \in G \text{ and } x \in A^G\,,$$

is called the (left) G-shift on A^G. Given a configuration $x \in A^G$ and a subset $F \subseteq G$, the element $x\restriction_F \in A^F$ defined by $x\restriction_F(g) = x(g)$ for all $g \in F$ is called the *restriction* of x to F. For any subset $G' \subseteq G$, we let $\pi_{G'} : A^G \to A^{G'}$ be the map such that $\pi(x) = x\restriction_{G'}$, for all $x \in A^G$.

Definition 5.7. *A* cellular automaton *over the group G and the alphabet A is a map $\phi : A^G \to A^G$ satisfying the following condition: there exist a finite subset $F \subseteq G$ and a map $\alpha : A^F \to A$ such that*

$$\phi(x)(g) = \alpha((g^{-1}x)\restriction_F) \tag{5.1}$$

for all $x \in A^G$ and $g \in G$. In this case, F is a memory set *of ϕ and α is the* local defining map *for ϕ associated with F.*

Definition 5.8. *Let R be a ring, let N be a left R-module, let G be a group and consider a cellular automaton $\phi : N^G \to N^G$. We say that ϕ is a* linear cellular automaton *if there is a memory set F and a local defining map $\alpha : N^F \to N$ that is a homomorphism of left R-modules.*

The following lemma is a particular case of [13, Theorem 1.9.1].

Lemma 5.9. *Let R be a ring, let N be a left R-module, let G be a group and consider a map $\phi : N^G \to N^G$. Endow N^G with the product of the discrete topologies on each copy of N. The following are equivalent:*

(1) *ϕ is a linear cellular automaton;*

(2) *ϕ is a continuous G-equivariant homomorphism.*

In this subsection we use the general results we proved for qframes to deduce a surjunctivity theorem for a suitable family of linear cellular automaton. Let us start defining a natural qframe associated with strictly linearly compact modules (see Definition A.3).

Definition 5.10. *Let R be a discrete ring and let M be a linearly topologized left R-module. We let $(\mathcal{N}(M), \leqslant)$ be the poset of submodules of M, ordered by* reverse *inclusion.*

Lemma 5.11. *Let R be a discrete ring, let M and N be strictly linearly compact left R-modules and let $\phi : M \to N$ be a continuous homomorphism of left R-modules. Then, $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are qframes and the map*

$$\Phi : \mathcal{N}(N) \to \mathcal{N}(M) \quad \textit{such that} \quad \Phi(C) = \phi^{-1}(C)$$

is a homomorphism of qframes. Furthermore, if ϕ is injective then Φ is surjective and algebraic, and, under these hypotheses, Φ is injective if and only if ϕ is surjective.

Proof. It is easy to check that $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are complete lattices (in fact, the maximum of $\mathcal{N}(M)$ is 0, while its minimum is M; furthermore the meet of two closed submodules is the closure of their sum, while the join of a family (finite or infinite) of closed submodules is their intersection). To show that $\mathcal{N}(M)$ is modular take A, B, $C \in \mathcal{N}(M)$ such that $A \leqslant C$ (that is, $C \subseteq A$). Using, the modularity of the lattice of all submodules $\mathcal{L}(M)$ of M with the usual order, one gets $C + (B \cap A) = (C + B) \cap A$, thus

$$\overline{C + (B \cap A)} = \overline{(C + B) \cap A} = \overline{(C + B)} \cap A\,,$$

which is the modular law in $\mathcal{N}(M)$. The fact that $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are upper continuous is proved for example in [29, Theorem 28.20].

The map Φ is well-defined by the continuity of ϕ, that ensures that $\phi^{-1}(C) \in \mathcal{N}(M)$, for all $C \in \mathcal{N}(N)$. Since ϕ^{-1} commutes with arbitrary intersections, Φ commutes with arbitrary joins. Let now $C_1 \leqslant C_2 \in \mathcal{N}(N)$ and let us show that $\Phi([C_1, C_2]) = [\Phi(C_1), \Phi(C_2)]$. Indeed, given $C \in [\Phi(C_1), \Phi(C_2)]$, $\phi^{-1}(C_2) \subseteq C \subseteq \phi^{-1}(C_1)$, so that $C_2 \cap \phi(M) \subseteq \phi(C) \subseteq C_1 \cap \phi(M)$. Thus,

$$\begin{aligned} C &= \Phi(\phi(C)) = \Phi(\overline{\phi(C) + (C_2 \cap \phi(M))}) \\ &= \Phi(\overline{(\phi(C) + C_2)} \cap \phi(M)) = \Phi(\overline{\phi(C) + C_2})\,, \end{aligned}$$

where in the first line we used that C contains the kernel of ϕ, while in the second line we applied the modular law. Since $\overline{\phi(C) + C_2} \in [C_1, C_2]$, Φ sends segments to segments and so it is a morphism of qframes.

Suppose now that ϕ is injective. To show that Φ is surjective notice that, by the injectivity of ϕ, $\Phi([0, 1]) = [0, \Phi(1)] = [0, \mathrm{Ker}(\phi)] = \mathcal{N}(M)$. It remains to show that Φ is algebraic: it is enough to notice that $\mathrm{Ker}(\Phi) = \phi(M)$ and that, given C_1, $C_2 \in [\phi(M), 1]$ such that $\Phi(C_1) = \Phi(C_2)$, then

$$C_1 = C_1 \cap \phi(M) = \phi(\phi^{-1}(C_1)) = \phi(\phi^{-1}(C_2)) = C_2 \cap \phi(M) = C_2\,.$$

Finally, since Φ is algebraic, Φ is injective if and only if $\mathrm{Ker}(\Phi) = 0$, that is, $\phi(M) = M$, which is equivalent to say that ϕ is surjective. ☐

Theorem 5.12. *Let R be a ring, let G be a sofic group and let ${}_RN$ be an Artinian left R-module. Then, any linear cellular automaton $\phi : N^G \to N^G$ is surjunctive.*

Proof. Suppose that $\phi : N^G \to N^G$ is an injective linear cellular automaton and let us prove that it is surjective.

By Lemmas A.5 and A.6 (2), N^G is strictly linearly compact so, by Lemma 5.11, $\mathcal{N}(N^G)$ is a qframe. Furthermore, the map

$$\rho : G \to \mathrm{Aut}(\mathcal{N}(N^G)) \quad \rho(g) = \rho_g\,,$$

such that $\rho_g(K) = \lambda_g^{-1}(K)$, for all $K \in \mathcal{N}(N^G)$ and $g \in G$, is a right action and the map

$$\Phi : \mathcal{N}(N^G) \to \mathcal{N}(N^G) \quad \Phi(K) = \phi^{-1}(K)\,,$$

for all $K \in \mathcal{N}(N^G)$, is a G-equivariant surjective algebraic homomorphism of qframes.

Let $y = \pi_e^{-1}(\{0\})$, where $\pi_e : N^G \to N^e$ is the usual projection, notice that $[0, y] \cong \mathcal{N}(N)$ is a Noetherian lattice and let $y_g = \rho_g(y)$, for all $g \in G$. It is clear that $\{y_g : g \in G\}$ is a basis for $\mathcal{N}(N^G)$.

By the above discussion, hypotheses (a_*) and (b_*) of Theorem 4.7 are satisfied and so Φ is injective. By Lemma 5.11, ϕ is surjective. ☐

Corollary 5.13. *Let D be a division ring, the V be a finite dimensional left vector space over D, let G be a group and let $\phi : V^G \to V^G$ be a linear cellular automaton. If G is either (finite-by-polycyclic)-by-sofic or free-by-sofic, then ϕ is surjunctive.*

Proof. The result follows by the duality between strictly linearly compact and discrete vector spaces (see the Duality Theorem in the Appendix), Corollary 5.5 and Corollary 5.4. $\square$

Let us remark that, by the same argument of the above corollary, a positive solution to Question 5.6 would solve also the L-Surjunctivity Conjecture.

A Topological modules and duality

Let R be a topopological ring (addition and multiplication are continuous functions $R \times R \to R$), let M be a left R-module and let τ be a topology on M. The pair (M, τ) is said to be a *topological module* if it is a topological group and the scalar multiplication $R \times M \to M$ is a continuous map.

Definition A.1. *Let G be a group, let R be a topological ring, let N be a topological left R-module and consider a cellular automaton $\phi : N^G \to N^G$. We say that ϕ is a* linear cellular automaton *if there is a memory set F and a local defining map $\alpha : N^F \to N$ that is a continuous homomorphism of left R-modules (where N^F is endowed with the product topology).*

Lemma A.2. *Let G be a group, let R be a topological ring, let N be a topological left R-module and consider a map $\phi : N^G \to N^G$. Endow N^G with the product topology and consider the following statements:*

(1) *ϕ is a linear cellular automaton;*

(2) *ϕ is a continuous and G-equivariant homomorphism.*

Then, (1)⇒(2). If N is discrete, then also (2)⇒(1).

Proof. (1)⇒(2). Let $F \subseteq G$ be a memory set and let $\alpha : N^F \to N$ be the associated local defining map. For any subset $G' \subseteq G$ we let $\pi : N^G \to N^{G'}$ be the canonical projection $\pi(x) = x\restriction_{G'}$. Recall that a typical basic neighborhood of 0 for the product topology on N^G is of the form $\pi_{G'}^{-1}(A)$ for some finite subset $G' \subseteq G$ and some open neighborhood A of 0 in $N^{G'}$.

For any open neighborhood A of 0 in N, $\phi^{-1}(\pi_{\{g\}}^{-1}(A)) = \pi_{gF}^{-1}(\alpha^{-1}(A))$ is an open neighborhood in N^G. This is enough to show that ϕ is continuous since $\{\pi_{\{g\}}^{-1}(A) : g \in G\}$ is a prebase of the topology. It is not difficult to show that ϕ is G-equivariant.

(2)⇒(1). When N is discrete this follows as in [13, Theorem 1.9.1] □

A.1 (Strictly) Linearly compact modules

From now on, we fix a discrete ring R (i.e., a topological ring endowed with the discrete topology). A topological left R-module (M, τ) is *linearly topologized* if the filter of neighborhoods of 0 has a filter base consisting of open submodules, that is, there exists a family of open submodules $\mathcal{B} = \{B_i\}_{i \in I}$ such that a subset $S \subseteq M$ is a neighborhood of 0 if and only if it contains B_i, for some $i \in I$. We call $\mathcal{B}$ a *linear base* of τ. In this setting, (M, τ) is a Hausdorff space if and only if 0 is a closed point, that is, $\bigcap_{i \in I} B_i = \{0\}$.

Let (M_1, τ_1) and (M_2, τ_2) be linearly topologized left R-modules and let $\phi : M_1 \to M_2$ be a homomorphism of left R-modules. Then, ϕ is continuous if and only if $\phi^{-1}(B)$ is an open neighborhood of M_1, for any open neighborhood B, belonging to a fixed base for the filter of neighborhoods of 0 in M_2. We denote by $\mathrm{CHom}_R(M_1, M_2)$ the group of continuous homomorphisms from M_1 to M_2.

Definition A.3. *Let R be a ring. We denote by R-*LT *the category of linearly topologized Hausdorff left R-modules and continuous homomorphisms.*

Let $(M, \tau) \in R$-LT. A *(open, closed) linear variety* is a subset of M of the form $x + N$ where N is a (open, closed) submodule. Given a set I and a family $\mathcal{F} = \{x_i + N_i : i \in I\}$ of linear varieties, $\mathcal{F}$ has the *finite intersection property* (f.i.p.) if $\bigcap_{i \in J} x_i + N_i \neq \varnothing$, for any finite subset $J \subseteq I$.

Definition A.4. *Let $(M,\tau) \in R$-LT. Then,*

(1) *(M,τ) is* linearly compact *if any family $\mathcal{F}$ of open linear varieties with the f.i.p. has non-empty intersection;*

(2) *(M,τ) is* strictly linearly compact *if it is linearly compact and, given any $(M',\tau') \in R$-LT and any surjective continuous homomorphism $\phi : M \to M'$, ϕ is open.*

In the following example we work out the above definition in the discrete case.

Example A.5. Let (M,τ) be a discrete module. It is not difficult to show that if M is Artinian, then it is discrete if and only if it is Hausdorff. Furthermore, if τ is the discrete topology, then M is Artinian if and only if M is strictly linearly compact.

The proof of the following properties can be found in [29, Chapter VII].

Proposition A.6. *Let R be a ring and let $(M,\tau) \in R$-LT.*

(1) *M is (strictly) linearly compact if and only if both N and M/N are (strictly) linearly compact (with respect to the induced topologies), for any closed $N \leqslant M$.*

(2) *If M is the product of a family $\{(N_i,\tau_i) : i \in I\}$, then M is (strictly) linearly compact if and only if N_i if (strictly) linearly compact for all $i \in I$;*

(3) *M is (strictly) linearly compact if and only if M is complete and M/B_i is (strictly) linearly compact discrete, where $\mathcal{B} = \{B_i : i \in I\}$ is a linear base for M.*

If R is a field, by part (3) of the above proposition, a linearly topologized Hausdorff R-vector space is linearly compact if and only if it is strictly linearly compact, if and only if it is complete and it has a base of neighborhoods made of vector subspaces of finite codimension.

We will need also the following fact, which can be found again in [29, Chapter VII]:

Lemma A.7. *Let (M_1,τ_1) and $(M_2,\tau_2) \in R$-LT. If M_1 is (strictly) linearly compact and $\phi : M_1 \to M_2$ is a continuous morphism, then $\phi(M_1)$ is (strictly) linearly compact.*

Definition A.8. *Let R be a ring. We denote by R-SLC the full subcategory of R-LT whose objects are the strictly linearly compact modules.*

A.2 Duality

We start fixing the setting that we will maintain all along this section.

(Dual.1) R is a ring that is linearly compact as a left R-module endowed with the discrete topology;

(Dual.2) ${}_RK$ is a minimal injective cogenerator, that is, ${}_RK$ is the injective envelope of the direct sum of a family of representatives of the simple left R-modules. We assume ${}_RK$ is Artinian;

(Dual.3) we denote by A the endomorphism ring of ${}_RK$.

Example A.9. The above setting for duality happens, for example, when R is a (skew) field or a commutative local complete Noetherian ring (see [22]).

We define two contravariant functors:

$$\mathrm{CHom}_R(-, K) = (-)^* : R\text{-SLC} \to \text{Mod-}A\,, \quad \mathrm{Hom}_A(-, K) = (-)^* : \text{Mod-}A \to R\text{-SLC} \qquad \text{(A.1)}$$

where, given a left A module N, the right R-module $N^* = \mathrm{Hom}_A(A, K)$ is endowed with the *finite topology*, that is, we take the following submodules as basic neighborhoods of 0:

$$\mathcal{V}(F) = \{f \in N^* : f(x) = 0\,,\ \forall x \in F\} \quad \text{for a finite subset } F \subseteq N\,.$$

The following result can be deduced by the main results of [24] and [25].

Duality Theorem. *Let R be a ring, let ${}_RK$ be a minimal injective cogenerator and let $A = \mathrm{End}_R(K)$. Suppose that R is linearly compact discrete and that ${}_RK$ is Artinian. Then, the above functors* (A.1) *define a duality between* Mod-A *and* R-SLC.

The above theorem is a particular case of the results discussed in [25], that was also generalized by many authors (see for example the bibliography of [23]). The particular statement above is enough for our needs and it allows us not to define "canonical choices" of topologies.

Remark A.10. The above Duality Theorem can be used to recover Sections 4 and 5 in [16]. In particular, the weak exactness of the duality functors described in [16, Section 5] can be improved to real exactness.

A.3 Applications

We state the following definition for a general category $\mathcal{C}$ but, in what follows, $\mathcal{C}$ will always be Mod-A or R-SLC for some rings R and A.

Definition A.11. *Let $\mathcal{C}$ be a category and let G be a group. A* left *(resp.,* right*)* representation *of G on $\mathcal{C}$ is a (anti)homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(C)$ for some object $C \in \mathcal{C}$. A* homomorphism $\phi : \mu_1 \to \mu_2$ *between two left (resp., right) representations $\mu_1 : G \to \mathrm{Aut}_{\mathcal{C}}(C_1)$ and $\mu_2 : G \to \mathrm{Aut}_{\mathcal{C}}(C_2)$ is a G-equivariant morphism $\phi : C_1 \to C_2$ in $\mathcal{C}$. We denote by* $\mathrm{lRep}(G, \mathcal{C})$ *and* $\mathrm{rRep}(G, \mathcal{C})$ *respectively the categories of left and right representations of G on $\mathcal{C}$.*

It is a classical observation that $\mathrm{rRep}(G, \text{Mod-}A)$ is canonically isomorphic to Mod-$A[G]$. Notice also that, by Lemma A.2 and Proposition A.6, a linear cellular automaton whose alphabet is a discrete Artinian left R-module is a morphism in $\mathrm{lRep}(G, R\text{-SLC})$.

Let $N \in R$-LT and endow N^G with the product topology. A *subshift* of N^G is a closed G-invariant submodule.

Corollary A.12 (Closed Image Property)**.** *Let G be a group and let R be a ring. Let $\lambda_1 : G \to \mathrm{Aut}_{R\text{-SLC}}(N_1)$ and $\lambda_2 : G \to \mathrm{Aut}_{R\text{-SLC}}(N_2)$ be two left representations of G on strictly linearly compact left R-modules. Given a morphism of representations $\phi : N_1 \to N_2$, the image $\phi(N_1)$ is closed and invariant under the action of G on N_2.*
In particular, given $N \in R$-SLC and a linear cellular automaton $\phi : N^G \to N^G$, the image of ϕ is a subshift.

Proof. Apply Lemma A.7. □

The following corollary of the Duality Theorem provides a "bridge" between automata and homomorphisms of right $A[G]$-modules.

Corollary A.13. *Let G be a group and consider the setting described in (Dual.1, 2, 3). The duality described in the Duality Theorem induces a duality between* Mod-$A[G]$ *and* $\mathrm{lRep}(G, R\text{-SLC})$.

Proof. It is enough to notice that a right action $\rho : G \to \mathrm{Aut}_A(M)$ of G on a right A-module M corresponds to a left action $\rho^* : G \to \mathrm{Aut}_{R\text{-SLC}}(M^*)$ of G on the dual module $M^* \in R$-SLC (just letting $\rho^*(g) = (\rho(g))^*$ for all $g \in G$) and that a left action $\lambda : G \to \mathrm{Aut}_{R\text{-SLC}}(N)$ on a strictly linearly compact left R-module N (notice that G acts via topological automorphisms) corresponds to a right action $\lambda^* : G \to \mathrm{Aut}_A(N^*)$ of G on $N^* \in$ Mod-A. To conclude one applies the Duality Theorem. □

Let $N \in R$-LT and consider a subshift $X \subseteq N^G$. A G-equivariant continuous morphism $\phi : X \to X$ is *reversible* if there is a continuous G-equivariant morphism $\psi : X \to X$ such that $\psi\phi = id_X$. The following corollary generalizes [13, Theorem 8.12.1]

Corollary A.14 (Strong Reversibility)**.** *Let G be a group, let R be a ring and let $N \in R$-SLC. Let $X \subseteq N^G$ be a subshift. Then, any bijective G-equivariant continuous morphism $\phi : X \to X$ is reversible.*
Furthermore, in the setting described in (Dual.1, 2, 3) and letting $H = K^n$ for some positive integer n, any injective linear cellular automaton $\phi : H^G \to H^G$ is reversible.

Proof. By Proposition A.6, X is strictly linearly compact, so ϕ is a topological automorphism and thus its inverse $\psi : X \to X$ is automatically a topological automorphism. The fact that ψ is G-equivariant can be deduced from the fact that it is the inverse of a G-equivariant map.
For the second part, just notice that the dual of H^G is the projective right $A[G]$-module $A[G]^n$ and so H^G is an injective object in $\mathrm{lRep}(G, R\text{-SLC})$. □

The following corollary improves [10, Theorem 1.3].

Corollary A.15 (L-Surjunctivity vs Stable finiteness)**.** *Let G be a group, consider the setting described in (Dual.1, 2, 3), let $\lambda : G \to \mathrm{Aut}_{R\text{-SLC}}(N) \in \mathrm{lRep}(G, R\text{-SLC})$ and let $M \in$ Mod-$A[G]$. There is an anti-isomorphism of rings*

$$\begin{aligned} \mathrm{End}_{\mathrm{lRep}(G,R\text{-SLC})}(N) &\longrightarrow \mathrm{End}_{A[G]}(M) \\ \phi &\longmapsto \phi^* . \end{aligned}$$

In particular, $\mathrm{End}_{\mathrm{lRep}(G,R\text{-SLC})}((K^n)^G)$ *is anti-isomorphic to* $\mathrm{Mat}_n(A[G])$ *for any positive integer n. Hence, $A[G]$ is stably finite if and only if any linear cellular automaton $\phi : (K^n)^G \to (K^n)^G$ is surjunctive, for any positive integer n.*

Proof. The first statement is an easy consequence of duality, while the fact that $\mathrm{End}_{\mathrm{lRep}(G,R\text{-SLC})}(K^n)$ is anti-isomorphic to $\mathrm{Mat}_k(A[G])$ follows noticing that the dual of $(K^n)^G$ is exactly $A[G]^n$ and that $\mathrm{End}_{A[G]}(A[G]^n) \cong \mathrm{Mat}_n(A[G])$. The last statement follows by the previous one recalling that linear cellular automata $(K^n)^G \to (K^n)^G$ are exactly the continuous G-equivariant endomorphisms of $(K^n)^G$ and using the second part of Corollary A.14. □

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